

RANK THREE GEOMETRY AND POSITIVE CURVATURE

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ABSTRACT. An axiomatic characterization of buildings of type C_3 due to Tits is used to prove that any cohomogeneity two polar action of type C_3 on a positively curved simply connected manifold is equivariantly diffeomorphic to a polar action on a rank one symmetric space. This includes two actions on the Cayley plane whose associated C_3 type geometry is not covered by a building.

The rank (or size) of a Coxeter matrix M coincides with the number of generators of its associated Coxeter system. The basic objects in Tits' local approach to buildings [Ti2] are the so-called chamber systems \mathcal{C} of type M (see also [Ro]). Indeed, if any so-called (spherical) residue (subchamber system) of \mathcal{C} of rank 3 is covered by a building, so is \mathcal{C} .

Recall that a polar G action on a Riemannian manifold M is an isometric action with a so-called section Σ , i.e., an immersed submanifold of M that meets all G orbits orthogonally. Since the action by the identity component of G is polar as well, we assume throughout without stating it that G is connected.

It is a key observation of [FGT] that the study of polar G actions on 1-connected positively curved manifolds M in essence is the study of a certain class of (connected) chamber systems $\mathcal{C}(M; G)$. Moreover, when the universal (Tits) cover of $\mathcal{C}(M; G)$ is a building it has the structure of a compact spherical building in the sense of Burns and Spatzier [BSp]. This was utilized in [FGT] to show:

THEOREM A. *Any polar G action of cohomogeneity at least two on a simply connected closed positively curved manifold M is equivariantly diffeomorphic to a polar G action on a rank one symmetric space if the associated chamber system $\mathcal{C}(M; G)$ is not of type C_3 .*

We note here, that when the action has no fixed points, the rank of $\mathcal{C}(M; G)$ is $\dim(M/G) + 1$, i.e., *one more* than the *cohomogeneity* of the action. In the above theorem the Cayley plane emerges only in cohomogeneity two and when G has fixed points. Moreover, there are indeed chamber systems with type $M = C_3$ whose universal cover is NOT a building (see, e.g., [Ne], [FGT], [Ly], [KL] and below). In our case, a polar G action on M is of type C_3 if and only if its orbit space M/G is a geodesic 2-simplex with angles $\pi/2, \pi/3$ and $\pi/4$.

Our aim here is to take care of this exceptional case and prove

THEOREM B. *Any polar G action on a simply connected positively curved manifold M of type C_3 is equivariantly diffeomorphic to a polar action on a rank one symmetric space. This*

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includes two actions on the Cayley plane where the universal covers of the associated chamber systems are not buildings.

Combining these results of course establishes, the

COROLLARY. *Any polar \mathbf{G} action of cohomogeneity at least two on a simply connected closed positively curved manifold M is equivariantly diffeomorphic to a polar \mathbf{G} action on a rank one symmetric space.*

This is in stark contrast to the case of cohomogeneity one, where in dimensions seven and thirteen there are infinitely many non-homogeneous manifolds (even up to homotopy). The classification work in [GWZ] also lead to the discovery and construction of a new example of a positively curved manifold (see [De] and [GVZ]).

By necessity, as indicated above, the proof of Theorem B is entirely different from the proof of Theorem A. In general, the geometric realization of our chamber systems $\mathcal{C}(M; \mathbf{G})$ utilized in the proof of Theorem A are not simplicial. However, in [FGT] it was proved that in fact

THEOREM C. *The geometric realization $|\mathcal{C}(M, \mathbf{G})|$ of a chamber system $\mathcal{C}(M, \mathbf{G})$ of type A_3 or C_3 associated with a simply connected polar \mathbf{G} -manifold M is simplicial.*

When the geometric realization of a chamber system of type M is simplicial it is called a *Tits geometry* of type M . This allows us to use an axiomatic characterization of C_3 geometries that are buildings (see [Ti2], Proposition 9). So rather than considering the universal cover $\tilde{\mathcal{C}}(M; \mathbf{G})$ directly, we construct in all but two cases a suitable cover of $\mathcal{C}(M; \mathbf{G})$ (possibly $\mathcal{C}(M; \mathbf{G})$ itself), and prove that it satisfies the C_3 building axiom of Tits. The two cases where this methods fails, are then recognized as being equivalent to two C_3 type polar actions on the Cayley plane $\mathbb{O}\mathbb{P}^2$ (cf. [PTh, GK]).

We note, that since all our chamber systems $\mathcal{C}(M, \mathbf{G})$ are homogeneous and those of type C_3 (and A_3) are Tits geometries an independent alternate proof of Theorem B follows from [KL].

1. PRELIMINARIES

The purpose of this section is threefold. While explaining the overall approaches to the strategies needed in the proof of Theorem B, we recall the basic concepts and establish notation.

Throughout \mathbf{G} denotes a compact connected Lie group acting on a closed 1- connected positively curved manifold M in a polar fashion and of type C_3 .

Fix a *chamber* C in a section Σ for the action. Then C is isometric to the orbit spaces M/\mathbf{G} and Σ/W , where W is the reflection group of Σ and W acts simply transitively on the chambers of Σ . Since the action is of type C_3 , C is a convex positively curved 2-simplex with geodesic sides = faces, ℓ_r, ℓ_t and ℓ_q opposite its vertices r, t and q with angles $\pi/2, \pi/3$ and $\pi/4$ respectively.

By the *Reconstruction Theorem* of [GZ] recall that any polar \mathbf{G} manifold M is completely determined by its so-called *polar data*. In our case, this data consist of \mathbf{G} and all its isotropy groups, *together with their inclusions* along a chamber C (cf. also Lemma 1.5 in [Go]). We denote the principal isotropy group by H , and the isotropy groups at vertices and opposite faces by $\mathbf{G}_r, \mathbf{G}_t, \mathbf{G}_q$ and $\mathbf{G}_{\ell_r}, \mathbf{G}_{\ell_t}, \mathbf{G}_{\ell_q}$ respectively. What remains after removing \mathbf{G} from this data will be referred to as the *local data* for the action.

With two exceptions, it turns out that only partial data are needed to show that the action indeed is equivalent to a polar action on a rank one symmetric space. Since the data in the two exceptional cases coincide with those of the exceptional C_3 actions on the Cayley plane, this will then complete the proof of Theorem A. In addition, it is worth noting, that since the groups G derived from those data (in 7.2 and 8.1) are maximal connected subgroups of F_4 , the identity component of the isometry group of the Cayley plane \mathbb{OP}^2 , their actions are uniquely determined and turn out to be polar.

The proof of Theorem A in all but the two exceptional cases is based on showing that the universal cover, $\tilde{\mathcal{C}}$ of the *chamber system* $\mathcal{C} = \mathcal{C}(M, G)$ associated to the polar action is a spherical Tits building [FGT]. Here, the homogeneous chamber system $\mathcal{C}(M, G)$ is the union $\bigcup_{g \in G} gC$ of all chambers with three *adjacency relations* one for each face: Specifically $g_1 C$ and $g_2 C$ are i adjacent if their respective i faces are the same in M . This chamber system with the thin topology, i.e., induced from the its path metric is a simplicial complex by Theorem C, and hence $\mathcal{C}(M, G)$ is a so-called C_3 geometry.

As indicated, the Fundamental Theorem of Tits used in [FGT] to show that $\tilde{\mathcal{C}}$ is a building yields nothing for rank three chamber systems as well as rank three geometries. Instead we will show that \mathcal{C} , or a cover we construct of \mathcal{C} is a C_3 building (and hence simply connected) by verifying an *axiomatic incidence characterization* (see section 3) of such buildings due also to Tits.

The *construction of chamber system covers* we utilize is equivalent in our context to the *principal bundle construction* of [GZ] (Theorem 4.5) for Coxeter polar actions and manifolds. Specifically for our case:

- Given the data, $H, G_{\ell_i}, G_j, i, j \in \{t, r, q\}$ and G for (M, G) , the data for $(P, L \times G)$ consists of graphs $\hat{H}, \hat{G}_{\ell_i}, \hat{G}_j$ in $\hat{G} := L \times G$ of compatible homomorphisms from $H, G_{\ell_i}, G_j, i, j \in t, r, q$ to L . In particular, the local data for $(P, L \times G)$ are isomorphic to the local data for (M, G) .
- Clearly L acts freely as a group of automorphisms, and $\mathcal{C}(P, L \times G)/L = \mathcal{C}(M, G)$, i.e., $\widehat{\mathcal{C}}(M; G) := \mathcal{C}(P, L \times G)$ is a chamber system covering of $\mathcal{C}(M; G)$.

In our case L will be S^1 (or in one case S^3).

2. BASIC TOOLS AND OBSTRUCTIONS

The aim of this section is to establish a number of properties and restrictions of the data to be used throughout. Unless otherwise stated G will be a compact connected Lie group and M a closed simply connected positively curved manifold.

Without any curvature assumptions we have the possibly well known

LEMMA 2.1 (Orbit equivalence). *Let M be a simply connected polar G manifold. Then the slice representation of any isotropy group is orbit equivalent to that of its identity component.*

Proof. Recall that the slice representation of an isotropy group $K = G_p \subset G$ restricted to the orthogonal complement T_p^\perp of the fixed point set of K inside the normal space to the orbit Gp is a polar representation. Clearly the finite group K/K_0 acts isometrically on the orbit space $\mathbb{S}(T_p^\perp)/K_0$, which is isometric to a chamber C of the polar K_0 action on the sphere $\mathbb{S}(T_p^\perp)$. Since C is convex with non-empty boundary its soul point (the unique point at maximal distance to

the boundary) is fixed by K/K_0 . This soul point, however, corresponds to a principal K_0 orbit, and hence to an exceptional K orbit unless K/K_0 acts trivially on C . However, by Theorem 1.5 [AT] there are no exceptional orbits of a polar action on a simply connected manifold. \square

Because of this, when subsequently talking casually about a slice representation we refer to the slice representation of its identity component unless otherwise stated.

Using positive curvature the following basic fact was derived in [FGT], Theorem 3.2:

LEMMA 2.2 (Primitivity). *The group G is generated by the (identity components) of the face isotropy groups of any fixed chamber.*

Naturally, the slice representations of G_t , G_q and G_r play a fundamental role. We denote the respective kernels of these representations by K_t, K_q and K_r and their quotients by \bar{G}_t, \bar{G}_q and \bar{G}_r . Since in particular the slice representation of G_t is of type A_2 it follows that the multiplicity triple of the polar G manifold M , i.e, the dimensions of the unit spheres in the normal slices along the edges ℓ_q, ℓ_r, ℓ_t is $(d, d, k) \in \mathbb{Z}_+^3$, where $d = 1, 2, 4$ or 8 .

For the kernels K_t and K_q , which are usually large groups, we have:

LEMMA 2.3 (Slice Kernel). *Let M be a simply connected polar G -manifold of type C_3 . If G acts effectively, then the kernel K_t , respectively K_q acts effectively on the slices T_q^\perp and T_r^\perp , respectively T_t^\perp and T_r^\perp .*

Proof. Note that K_t fixes all sections through t since K_t acts trivially on the slice T_t^\perp . We must prove that $K_t \cap K_q = \{1\}$, $K_t \cap K_r = \{1\}$ and $K_q \cap K_r = \{1\}$. We consider only $K_t \cap K_q$, since the arguments for the remaining cases are similar.

Note that since G is assumed to act effectively on M , and $K_t \cap K_q$ is contained in the principal isotropy group, it suffices to prove that $K_t \cap K_q$ is normal in G . By the primitivity (see 2.2), $G = \langle p_q^{-1}(\bar{G}_{q,0}), p_t^{-1}(\bar{G}_{t,0}) \rangle$, where $p_q : G_q \rightarrow \bar{G}_q$ is the quotient homomorphism and $\bar{G}_{q,0}$ is the identity component of \bar{G}_q and similarly for p_t . Thus, it suffices to show that $K_t \cap K_q$ is normal in each of $p_t^{-1}(\bar{G}_{t,0})$ and $p_q^{-1}(\bar{G}_{q,0})$. In each case, assuming the effective vertex isotropy group is connected does not alter the proof only simplifies notation. Accordingly, we proceed to assume that \bar{G}_t is connected, i.e., $\bar{G}_t = \bar{G}_{t,0}$ and will show that $K_t \cap K_q$ is a normal subgroup of G_t .

Note that $K_t \cap K_q$ is a normal subgroup of K_t acting trivially on both the slices T_t^\perp and T_q^\perp .

By assumption the quotient map $G_{t,0} \subset G_t \rightarrow \bar{G}_t$ is surjective when restricted to the identity component $G_{t,0}$ of G_t . A finite central cover $\tilde{G}_{t,0}$ of $G_{t,0}$ is isomorphic to the product $\tilde{K}_{t,0} \times \tilde{\bar{G}}_t$ where $\tilde{K}_{t,0}$ is locally isomorphic to the identity component $K_{t,0}$ of K_t and $\tilde{\bar{G}}_t$ is locally isomorphic to \bar{G}_t . In particular, G_t contains a connected and closed subgroup $\pi(\tilde{\bar{G}}_t)$ covering \bar{G}_t , where $\pi : \tilde{G}_{t,0} \rightarrow G_{t,0}$ is the cover map. Moreover, every element of the subgroup $\pi(\tilde{\bar{G}}_t)$ commutes with the elements in $K_{t,0}$. On the other hand, for every $h \in \pi(\tilde{\bar{G}}_t)$, the conjugation by h gives rise to an element in the automorphism group $\text{Aut}(K_t)$ since K_t is normal, hence defines a homomorphism $\phi : \pi(\tilde{\bar{G}}_t) \rightarrow \text{Aut}(K_t)$. Since $\phi(\pi(\tilde{\bar{G}}_t))$ has a trivial image in $\text{Aut}(K_{t,0})$ under the forgetful homomorphism $\text{Aut}(K_t) \rightarrow \text{Aut}(K_{t,0})$, the group $\phi(\pi(\tilde{\bar{G}}_t))$ is finite, and hence trivial because $\phi(\pi(\tilde{\bar{G}}_t))$ is connected. This implies that the elements of $\pi(\tilde{\bar{G}}_t)$ commute with the elements of

K_t . Since $G_t = \langle K_t, \pi(\tilde{G}_t) \rangle$ and $K_t \cap K_q$ is normal in K_t , it then follows that $K_t \cap K_q$ is a normal subgroup of G_t .

As mentioned above, the same arguments show that $K_t \cap K_q$ is normal in $p_t^{-1}(\tilde{G}_{t,0})$ in case \tilde{G}_t is not connected. The same arguments also show that $K_t \cap K_q$ is normal in $p_q^{-1}(\tilde{G}_{q,0})$. \square

Remark 2.4. It turns out that in all cases \tilde{G}_t is connected. In fact, this is automatic whenever $d \neq 2$, since \tilde{G}_t acts transitively on a projective plane. Up to local isomorphism its identity component is one of the groups $SO(3)$, $SU(3)$, $Sp(3)$, or F_4 corresponding to $d = 1, 2, 4$ and 8 respectively, and the slice representation is its standard polar representation of type A_2 (see also Table 4.3). In view of the Transversality Lemma 2.5 below, G_t is connected whenever $k \geq 2$. In the $(2, 2, 1)$ case, the connectedness of G_r (again by Lemma 2.5) implies that also in this case \tilde{G}_t is connected (see Proposition 5.5).

The following simple topological consequence of transversality combined with the fact that the canonical deformation retraction of the orbit space triangle minus any side to its opposite vertex lifts to M (or alternatively of the work [Wie]) will also be used frequently:

LEMMA 2.5 (Transversality). *Given a multiplicity triple (d, d, m) . Then the inclusion maps $G / G_r \subset M$, $G / G_q \subset M$ and $G / G_{\ell_t} \subset M$ are d -connected, $G / G_{\ell_r} \subset M$, and $G / G_{\ell_t} \subset M$ are $\min\{d, m\}$ connected, and $G / G_t \subset M$ is m -connected.*

Recall here that a continuous map is said to be k -connected if the induced map between the i th homotopy groups is an isomorphism for $i < k$ and a surjection for $i = k$.

Another Connectivity Theorem [Wi3] (Theorem 2.1) using positive curvature *à la* Synge is very powerful:

LEMMA 2.6 (Wilking). *Let M be a positively curved n -manifold and N a totally geodesic closed codimension k submanifold. Then the inclusion map $N \rightarrow M$ is $n - 2k + 1$ connected.*

If in addition N is fixed by an isometric action of a compact Lie group K with principal orbit of dimension $m(K)$, then the inclusion map is $n - 2k + 1 + m(K)$ connected.

We conclude this section with two severe restrictions on G stemming from positive curvature.

The first follow from the well known *Synge type* fact, that an isometric T^k action has orbits with $\dim \leq 1$ in odd dimensions and 0 in even dimensions, when M has positive curvature (cf. [Su]). In particular, since G_q has maximal rank among the isotropy groups, and the Euler characteristic $\chi(G / G_q) > 0$ if and only if $\text{rk}(G) = \text{rk}(G_q)$ ([HS] page 248) we conclude

LEMMA 2.7 (Rank Lemma). *The dimension of M is even if and only if $\text{rk}(G) = \text{rk}(G_q)$, and otherwise $\text{rk}(G) = \text{rk}(G_q) + 1$.*

When adapting Wilking's *Isotropy Representation* Lemma 3.1 from [Wi2] for positively curved G manifolds to polar manifolds of type C_3 we obtain:

LEMMA 2.8 (Sphere Transitive Subrepresentations). *Let $L_i \triangleleft G_{\ell_i}$, $i \in \{q, r, t\}$ be a simple normal subgroup and U an irreducible isotropy subrepresentation of G / L_i . Then (U, L_i) is isomorphic to a standard defining representation. In particular, L_i acts transitively on the sphere $S(U)$.*

Proof. Let U be an irreducible isotropy subrepresentation of \mathbf{G}/L_i not isomorphic to a summand of the slice representation of L_i on T_i^\perp . By [Wi2], U is isomorphic to a summand of the isotropy representation of L_i^*/L_i , where L_i^* is a vertex isotropy group. On the other hand, the almost effective factor of L_i^* is well understood (cf. the tables 4.3 and 4.4), which are all the standard defining representation. The desired result follows. \square

3. THE C_3 BUILDING AXIOM

Recall that Tits has provided an axiomatic characterization of buildings of irreducible type M when the geometric realization $|\mathcal{C}|$ (\mathcal{C} with the thin topology) of the associated chamber system \mathcal{C} , is a *simplicial complex*. This characterization is given in terms of the *incidence geometry* associated with \mathcal{C} .

The purpose of this section is to describe this characterization when $M = C_3$ and translate it to our context.

Here, by definition

- Vertices $x, y \in |\mathcal{C}|$ are *incident*, denoted $x * y$, if and only if x and y are contained in a closed chamber of $|\mathcal{C}|$.

Clearly, the incidence relation (not an equivalence relation) is preserved by the action of \mathbf{G} in our case.

To describe the needed characterization we will use the following standard terminology:

- The *shadow* of a vertex x on the set of vertices of type $i \in I$, denoted $\text{Sh}_i(x)$, is the union of all vertices of type i incident to x .

Following Tits [Ti2], when $M = C_3$, we call the vertices of type q, r and t , *points*, *lines*, and *planes* respectively. We denote by Q, R and T the set of points, lines, and planes in $\mathcal{C}(M; \mathbf{G})$. Notice that \mathbf{G} acts transitively on Q, R and T . With this terminology the axiomatic characterization [Ti2] (cf. Proposition 9 and the proof of the C_3 case on p. 544) alluded to above states:

THEOREM 3.1 (C_3 Axiom). *A connected Tits geometry of type C_3 is a building if and only if the following axiom holds:*

- (LL) *If two lines are both incident to two different points, they coincide.*

Equivalently:

- *If $\text{Sh}_Q(r) \cap \text{Sh}_Q(r')$ has cardinality at least two, then $r = r'$.*

or:

- *For any $q, q' \in Q$, with $q \neq q'$, $\text{Sh}_R(q) \cap \text{Sh}_R(q')$ has cardinality at most one.*

In our case, if $r \in R$ and $q \in Q$ are incident, (LL) is clearly equivalent to

- For any $r' \in G_q(r)$, $r' \neq r$, we have $G_r(q) \cap G_{r'}(q) = q$

or,

- For any $q' \in G_r(q)$, $q' \neq q$, we have $G_q(r) \cap G_{q'}(r) = r$

We proceed to interpret (LL) in terms of the isotropy groups data. This will be used either directly for $\mathcal{C}(M; \mathbf{G})$ or for a suitably constructed cover $\tilde{\mathcal{C}}(M; \mathbf{G})$ as described at the end of section 1. For notational simplicity we will describe it here only for $\mathcal{C}(M; \mathbf{G})$ (for the general case see remark 3.5 below).

PROPOSITION 3.2. *If $\mathcal{C}(M; \mathbf{G})$ is a building of type \mathbf{C}_3 , then the following holds:*

★ *for any pair of different points $q, q' \in Q$ both incident to an $r \in R$, we have*

$$\mathbf{G}_q \cap \mathbf{G}_{q'} \subset \mathbf{G}_{rq} \cap \mathbf{G}_{rq'}$$

where \mathbf{G}_{rq} denotes the isotropy group of the unique edge between r and q (cf. Theorem C).

Proof. Note that every line in the orbit $\mathbf{G}_q \cap \mathbf{G}_{q'}(r)$ is incident to both q and q' . Axiom (LL) implies that the orbit contains only one line, r and hence $\mathbf{G}_q \cap \mathbf{G}_{q'} \subset \mathbf{G}_r$. Since $\mathcal{C}(M; \mathbf{G})$ is a building, we have $\mathbf{G}_r \cap \mathbf{G}_q = \mathbf{G}_{rq}$ and $\mathbf{G}_r \cap \mathbf{G}_{q'} = \mathbf{G}_{rq'}$. The desired result follows. \square

We will see that the condition ★ together with an assumption on a suitable *reduction* of the \mathbf{G} action implies that $\mathcal{C}(M; \mathbf{G})$ is a building of type \mathbf{C}_3 .

To describe the reduction, let $r \in R$ be a line, and let $\mathbb{S}_{r,Q}^\perp$ be the normal sphere in the summand in the slice T_r^\perp . Then the shadow of r in Q is $\exp(\frac{\pi}{4}\mathbb{S}_{r,Q}^\perp)$. Moreover, the isotropy group \mathbf{G}_r acts transitively on $\mathbb{S}_{r,Q}^\perp$.

Let $\mathbf{K}_{r,Q}$ denote the identity component of the kernel of the transitive \mathbf{G}_r action on $\mathbb{S}_{r,Q}^\perp$.

It is clear that the fixed point connected component $M^{\mathbf{K}_{r,Q}}$ (containing r) is a cohomogeneity one $\mathbf{N}_0(\mathbf{K}_{r,Q})$ submanifold of M , where $\mathbf{N}_0(\mathbf{K}_{r,Q})$ is the identity component of the normalizer $\mathbf{N}(\mathbf{K}_{r,Q})$ of $\mathbf{K}_{r,Q}$ in \mathbf{G} . The corresponding chamber system denoted $\mathcal{C}(M^{\mathbf{K}_{r,Q}})$ is a subcomplex of $\mathcal{C}(M) := \mathcal{C}(M; \mathbf{G})$ that inherits an incidence structure, which gives rise to a Tits geometry of rank 2.

LEMMA 3.3 (Reduction). *The connected chamber system $\mathcal{C}(M, \mathbf{G})$ of type \mathbf{C}_3 is a building if for any $r \in R$, the reduction $\mathcal{C}(M^{\mathbf{K}_{r,Q}})$ is a \mathbf{C}_2 -building and ★ holds.*

Proof. If not, by Axiom (LL) there are two points $q \neq q' \in Q$ which are both incident to two different lines $r, r' \in R$. By ★ we know that $\mathbf{G}_q \cap \mathbf{G}_{q'} \subset \mathbf{G}_{rq} \cap \mathbf{G}_{rq'}$ and $\mathbf{G}_q \cap \mathbf{G}_{q'} \subset \mathbf{G}_{r'q} \cap \mathbf{G}_{r'q'}$. Therefore, the configuration $\{rq, rq', r'q, r'q'\}$ is contained in the fixed point set $M^{\mathbf{G}_q \cap \mathbf{G}_{q'}}$. Since by definition clearly $\mathbf{K}_{r,Q}$ is a subgroup of $\mathbf{G}_q \cap \mathbf{G}_{q'}$, we have that $M^{\mathbf{G}_q \cap \mathbf{G}_{q'}} \subset M^{\mathbf{K}_{r,Q}}$. This implies that there is a length 4 circuit in the \mathbf{C}_2 building $\mathcal{C}(M^{\mathbf{K}_{r,Q}})$. A contradiction. \square

The following technical criterion will be more useful to us:

LEMMA 3.4 (\mathbf{C}_3 Building Criterion). *The connected chamber system $\mathcal{C}(M, \mathbf{G})$ is a building if for any $r \in R$, the reduction $\mathcal{C}(M^{\mathbf{K}_{r,Q}})$ is a \mathbf{C}_2 -building and the following Property (P) holds:*

(P) *For any $q \in \text{Sh}_Q(r)$, and any Lie group \mathbf{L} with $\mathbf{K}_{r,Q} \subset \mathbf{L} \subset \mathbf{G}_q$ but $\mathbf{L} \not\subset \mathbf{G}_{rq}$, the normalizer $\mathbf{N}(\mathbf{K}_{r,Q}) \cap \mathbf{L}$ is not contained in \mathbf{G}_{rq} either.*

Proof. By the previous lemma it suffices to verify ★. Suppose ★ is not true. Then there is an $r \in R$ and a pair of points $q \neq q'$ both incident to r such that $\mathbf{G}_q \cap \mathbf{G}_{q'}$ is not a subgroup of \mathbf{G}_{rq} . Let $\mathbf{L} = \mathbf{G}_q \cap \mathbf{G}_{q'}$. By Assumption (P), there is an $\alpha \in \mathbf{N}(\mathbf{K}_{r,Q}) \cap \mathbf{L}$ so that $\alpha \notin \mathbf{G}_{rq}$. However, $\mathbf{G}_r \cap \mathbf{G}_q \cap \mathbf{N}(\mathbf{K}_{r,Q}) = \mathbf{G}_{rq} \cap \mathbf{N}(\mathbf{K}_{r,Q})$ since $M^{\mathbf{K}_{r,Q}}$ is an \mathbf{C}_2 building. In particular, $\alpha \notin \mathbf{G}_r$, and so there is a length 4 circuit $\{rq, q\alpha(r), \alpha(r)q', q'r\}$ in the \mathbf{C}_2 building $\mathcal{C}(M^{\mathbf{K}_{r,Q}})$. A contradiction. \square

Remark 3.5. For an \mathbf{S}^1 cover $\tilde{\mathcal{C}} := \mathcal{C}(P, \mathbf{S}^1 \times \mathbf{G})$ of $\mathcal{C}(M, \mathbf{G})$ constructed as above note that the property ★ is inherited from (M, \mathbf{G}) . Likewise, the group $\hat{\mathbf{K}}$ being the graph of the

homomorphism $G_{\ell_t} \subset G_r$ to S^1 restricted to $K := K_{r,Q}$ satisfies Property (P) when K does. For this note that by construction the local data for the reduction P^K are isomorphic to the local data for M^K . It then follows as in the proofs above, that if a component of the reduction $\mathcal{C}(P^K) \subset \mathcal{C}(P, S^1 \times G)$ is a C_2 -building, then the corresponding component of $\tilde{\mathcal{C}}$ will be a C_3 building covering $\mathcal{C}(M, G)$, and our main result, Theorem 4.10, from the [FGT] applies.

Remark 3.6. If $K' = K'_{r,Q} \subset K_{r,Q} = K$ is a subgroup, then the assumption of $\mathcal{C}(M^K)$ being a C_2 building in the above criterion may be replaced by, the fixed point component $\mathcal{C}(M^{K'}) \supset \mathcal{C}(M^K)$ being a C_2 building, or a rank 3 building. For the latter, we notice that, by Charney-Lytchak [CL] Theorem 2, a rank 3 spherical building is a CAT(1) space, hence any two points of distance less than π are joined by a unique geodesic. This clearly excludes a length 4 circuit in the above proof, since its perimeter is π .

Remark 3.7. Note that clearly $K_t \subset K_{\ell_q} \cap K_{\ell_r}$ and similarly for the other kernels of vertex and edge isotropy groups. In particular, for the identity component K' of K_t we have $K' \subset K$, where $K (= K_{r,Q})$ is the identity component of the kernel of G_r acting on \mathbb{S}^d . Consequently, the reduction $M^{K'}$ is a cohomogeneity two manifold of type either A_3 , or C_3 containing the cohomogeneity one manifold M^K (cf. 3.6 above).

4. CLASSIFICATION OUTLINE AND ORGANIZATION

The subsequent sections are devoted to a proof of the following main result of the paper:

THEOREM 4.1. *Let M be a compact, simply connected positively curved polar G -manifold with associated chamber system $\mathcal{C}(M; G)$ of type C_3 . Then the universal cover $\tilde{\mathcal{C}}$ of $\mathcal{C}(M; G)$ is a building if and only if (M, G) is not equivariantly diffeomorphic to one of the exceptional polar actions on $\mathbb{O}P^2$ by $G = \mathrm{SU}(3) \cdot \mathrm{SU}(3)$ or $G = \mathrm{SO}(3) \cdot G_2$.*

This combined with the main result of [FGT] proves Theorem B in the introduction.

The purpose of this section is to describe how the proof is organized according to four types of scenarios driven by the possible compatible types of slice representations for G_t and G_q at the vertices t and q of a chamber C .

The common feature in each scenario and all cases is the determination of all local data. The basic input for this is indeed knowledge of the slice representations at the vertices t and q of a chamber C , and Lemma 2.3. The local data identifies the desired $K \subset G_r$ reduction M^K with its cohomogeneity one action by $N(K)$ referred to in the Building Criteria Lemma 3.4, with Property (P) being essentially automatic. The main difficulty is to establish that $\mathcal{C}(M^K) \subset \mathcal{C}(M; G)$ or the corresponding reduction in a cover (which by construction has the same local data) is a C_2 building. The first step for this frequently uses the following consequence of the classification work on positively curved cohomogeneity one manifolds in [GWZ] and [Ve].

LEMMA 4.2. *Any simply connected positively curved cohomogeneity one manifold with multiplicity pair different from $(1, 1)$, $(1, 3)$ and $(1, 7)$ is equivariantly diffeomorphic to a rank one symmetric space.*

As already pointed out and used, there are only four possible (effective) slice representations at t , in particular forcing the codimensions of the orbit strata corresponding to ℓ_q, ℓ_r , and ℓ_t to

be $d+1$, $d+1$ and $k+1$, where $d = 1, 2, 4$ or 8 . In Table 4.3, L^\pm , respectively H are the singular, respectively principal isotropy groups for the effective slice representation, χ by \bar{G}_t restricted to the unit sphere, and $l_\pm + 1$ are the codimensions of the singular orbits.

n	\bar{G}_t	χ	L^-	L^+	H	(l_-, l_+)	W
4	$SO(3)$		$S(O(2) O(1))$	$S(O(1) O(2))$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(1, 1)$	A_2
7	$PSU(3)$	Ad	$S(U(2) U(1))/\Delta(\mathbb{Z}_3)$	$S(U(1) U(2))/\Delta(\mathbb{Z}_3)$	T^2 / \mathbb{Z}_3	$(2, 2)$	A_2
13	$Sp(3)/\Delta(\mathbb{Z}_2)$	ψ_{14}	$Sp(2) Sp(1)/\Delta(\mathbb{Z}_2)$	$Sp(1) Sp(2)/\Delta(\mathbb{Z}_2)$	$Sp(1)^3/\Delta(\mathbb{Z}_2)$	$(4, 4)$	A_2
25	F_4	ψ_{26}	$Spin(9)$	$Spin(9)$	$Spin(8)$	$(8, 8)$	A_2

TABLE 4.3. Effective t -slice representations on $\mathbb{S}_t^\perp = \mathbb{S}^n$

Similarly (see Table 4.4), the identity component $(\bar{G}_q)_0$ of possible effective C_2 type slice representations at q which are compatible with the multiplicity restrictions in Table 4.3 are known as well (see e.g. Table E of [GWZ] in which we have corrected an error for the exceptional $SO(2) Spin(7)$ representation (see also [GKK] (Main Theorem)).

n	$(\bar{G}_q)_0$	χ	L^-	L^+	H	(l_-, l_+)	W
$8k + 15, k \geq 0$	$\frac{Sp(2) Sp(k+2)}{\Delta(\mathbb{Z}_2)}$	$\nu_2 \hat{\otimes} \nu_{k+2}$	$\frac{Sp(2) Sp(k)}{\Delta(\mathbb{Z}_2)}$	$\frac{Sp(1)^2 Sp(k+1)}{\Delta(\mathbb{Z}_2)}$	$\frac{Sp(1)^2 Sp(k)}{\Delta(\mathbb{Z}_2)}$	$(4, 4k + 3)$	C_2
$4k + 7, k \geq 1$ even	$\frac{SU(2) SU(k+2)}{\Delta(\mathbb{Z}_2)}$	$\mu_2 \hat{\otimes} \mu_{k+2}$	$\frac{\Delta SU(2) SU(k)}{\Delta(\mathbb{Z}_2)}$	$S^1 \cdot SU(k+1)$	$\frac{S^1 \cdot SU(k)}{\Delta(\mathbb{Z}_2)}$	$(2, 2k + 1)$	C_2
$4k + 7, k \geq 1$ odd	$SU(2) SU(k+2)$	$\mu_2 \hat{\otimes} \mu_{k+2}$	$\Delta SU(2) SU(k)$	$S^1 \cdot SU(k+1)$	$S^1 \cdot SU(k)$	$(2, 2k + 1)$	C_2
$4k + 7, k \geq 1$	$\frac{U(2) SU(k+2)}{\Delta(\mathbb{Z}_k)}$	$\mu_2 \hat{\otimes} \mu_{k+2}$	$\frac{\Delta U(2) SU(k)}{\Delta(\mathbb{Z}_k)}$	$\frac{T^2 \cdot SU(k+1)}{\Delta(\mathbb{Z}_k)}$	$\frac{T^2 \cdot SU(k)}{\Delta(\mathbb{Z}_k)}$	$(2, 2k + 1)$	C_2
7	$\frac{U(2) SU(2)}{\Delta(\mathbb{Z}_2)}$	$\mu_2 \hat{\otimes} \mu_2$	$\Delta SO(3)$	T^2	S^1	$(2, 1)$	C_2
$2k + 3, k \geq 1$ even	$\frac{SO(2) SO(k+2)}{\Delta(\mathbb{Z}_2)}$	$\rho_2 \hat{\otimes} \rho_{k+2}$	$\frac{\Delta SO(2) SO(k)}{\Delta(\mathbb{Z}_2)}$	$SO(k+1)$	$SO(k)$	$(1, k)$	C_2
$2k + 3, k \geq 1$ odd	$SO(2) SO(k+2)$	$\rho_2 \hat{\otimes} \rho_{k+2}$	$\Delta SO(2) SO(k)$	$\mathbb{Z}_2 \cdot SO(k+1)$	$\mathbb{Z}_2 \cdot SO(k)$	$(1, k)$	C_2
13	$SO(2) G_2$	$\rho_2 \hat{\otimes} \phi_7$	$\Delta SO(2) SU(2)$	$\mathbb{Z}_2 \cdot SU(3)$	$\mathbb{Z}_2 \cdot SU(2)$	$(1, 5)$	C_2
15	$\frac{SO(2) Spin(7)}{\Delta(\mathbb{Z}_2)}$	$\rho_2 \hat{\otimes} \Delta_7$	$\Delta SO(2) SU(3)$	G_2	$SU(3)$	$(1, 6)$	C_2
9	$SO(5)$	ad	$U(2)$	$SO(3) SO(2)$	T^2	$(2, 2)$	C_2
19	$SU(5)$	$\Lambda^2 \mu_5$	$Sp(2)$	$SU(2) SU(3)$	$SU(2)^2$	$(4, 5)$	C_2
	$U(5)$	$\Lambda^2 \mu_5$	$S^1 \cdot Sp(2)$	$S^1 \cdot SU(2) SU(3)$	$S^1 \cdot SU(2)^2$		

TABLE 4.4. Effective q -slice representation on $\mathbb{S}_q^\perp = \mathbb{S}^n$

Aside from a few exceptional representations, they are the isotropy representations of the Grassmannians $G_{2,m+2}(k)$ of 2-planes in k^{m+2} , where $k = \mathbb{R}, \mathbb{C}$, or \mathbb{H} . The pairs of multiplicities

that occur for the exceptional representations are $(1, 6), (1, 5), (4, 5), (2, 2)$, corresponding to $\bar{G}_q = \mathrm{SO}(2) \mathrm{Spin}(7), \mathrm{SO}(2) \mathrm{G}_2, \mathrm{SU}(5), \mathrm{U}(5)$, or $\mathrm{SO}(5)$.

Note that effectively, there are only four *exceptional* G_q slice representations, corresponding to the last four rows of Table 4.4. However, *special situations* occur also when the slice representation of \bar{G}_q is the isotropy representation of the real Grassmann manifold, when its multiplicity $(1, k)$ happen to have $k = d = 1, 2, 4$ or 8 . We will refer to these as *flips*. As may be expected, the low multiplicity cases $(1, 1, 1), (1, 1, 5)$ and $(2, 2, 3)$ play important special roles. The latter two are where the exceptional Cayley plane emerges, the only cases where complete information about the polar data are required.

Accordingly we have organized the proof of 4.1 into four sections depending on the type of slice representations we have along Q : Three Grassmann flips, three Grassmann series (two non minimal), two minimal Grassmann representations, and four exceptional representations.

5. GRASSMANN FLIP G_q SLICE REPRESENTATION

This section will deal with the multiplicity cases $(d, d, 1)$ with $d = 2, 4$ and 8 , leaving $d = 1$ (minimal and odd) for section 7. We have the following common features:

LEMMA 5.1. *The isotropy groups G_q and G_r are connected, and the reducible \bar{G}_r slice representation on $\mathbb{S}^1 * \mathbb{S}^d = \mathbb{S}^{d+2}$ is the standard action by $\mathrm{SO}(2) \mathrm{SO}(d+1)$. For the kernels of the slice representations we have that $K_t = \{1\}$, $K_q = K_{\ell_r}$ and $K_r = K_{\ell_q}$.*

Proof. The Transversality Lemma 2.5 implies that the orbits $Q = G_q$ and $R = G_r$ are simply connected since M is. In particular, G_q and G_r are connected since G is. The second claim follows since d is even (cf. Appendix in [FGT] for a description of reducible polar representations).

Since $(\bar{G}_q)_{\ell_r}$ (cf. Table 4.4) as well as $(\bar{G}_r)_{\ell_q}$ act effectively on the respective normal spheres \mathbb{S}^d , we see that $K_q = K_{\ell_r}$ and $K_r = K_{\ell_q}$. Also since $K_t \subset K_{\ell_r}$ we have $K_t \cap K_q = K_t$ but $K_t \cap K_q = \{1\}$ by the Kernel Lemma 2.3 and hence $K_t = \{1\}$. \square

Recall that K is the identity component of the kernel of the G_r action restricted to \mathbb{S}^d .

LEMMA 5.2. *Clearly $K \triangleleft G_r$, and $K \subset G_{\ell_t}$ acts transitively on the corresponding normal sphere \mathbb{S}^1 with kernel identity component of K_r . Moreover, $K \cap K_q = \{1\}$ and hence $K \subset G_q \rightarrow \bar{G}_q$ is injective.*

The reduction M^K is a positively curved irreducible cohomogeneity one $N_0(K)$ manifolds with multiplicity pair $(d, 1)$.

Proof. Note that $K \cap K_q$ acts trivially on $\mathbb{S}^1 * \mathbb{S}^d$, so $K \cap K_q \subset K_r$. The second claim follows since $K_r \cap K_q = \{1\}$.

Since $K \triangleleft G_{\ell_t} \rightarrow \bar{G}_q$ is injective, we see from Table 4.4 that $N(K) \cap G_q / G_{\ell_t} = N(K) \cap \bar{G}_q / \bar{G}_{\ell_t} = \mathbb{S}^1$, and hence M^K is cohomogeneity one with multiplicity pair $(d, 1)$.

To complete the proof assume by contradiction that the action is reducible, i.e., that the action by $N_0(K)/K$ on M^K is equivalent to the sum action of $\mathrm{SO}(2) \mathrm{SO}(d+1)$ on $\mathbb{S}^1 * \mathbb{S}^d$, where the isotropy $(N_0(K)/K)_q$ is $\mathrm{SO}(2) \mathrm{SO}(d)$. In all cases, it is easy to see that, the center of G_q intersects the center of $N_0(K)$ in a nontrivial subgroup \mathbb{S}^1 . This, together with primitivity implies that, \mathbb{S}^1

is in the center of G . Notice that, as a subgroup of G_q , S^1 can not be in K_q because $K_q \subset H$, and the factor $SO(2) \triangleleft \bar{G}_q$ acts freely on the unit sphere of the slice T_q^\perp . Thus, the fixed point set M^{S^1} coincides with the orbit $Gq = G/G_q$. From the classification of positively curved homogeneous spaces we get immediately that, G is the product of S^1 (or T^2 if $d = 2$) with one of a few orthogonal groups or unitary groups, each of which is not big enough to contain the simple group G_t . The desired result follows. \square

Although what remains is in spirit the same for all the flip cases, we will carry out the arguments for each case individually, beginning with $d = 8$.

PROPOSITION 5.3. *In the Flip (8,8,1) case, $\mathcal{C}(M, G)$ or an S^1 covering is a building, with the isotropy representation of $E_7 / E_6 \times S^1$ as a linear model.*

Proof. From Lemma 5.1 and Tables 4.4 and 4.3 we obtain the following information about the local data: $G_t = F_4 \supset Spin(8) = H$, $G_{\ell_q} = Spin(9)$, $G_{\ell_r} = Spin(9)$, $G_q = S^1 \cdot Spin(10) \supset \Delta(S^1) \cdot Spin(8) = G_{\ell_t}$, and $G_r = S^1 \cdot Spin(9)$.

Also $G_r \triangleright K = \Delta(S^1) \subset G_{\ell_t} \subset G_q$, and from Lemma 5.2 and Lemma 4.2 we see that the corresponding reduction, M^K is \mathbb{S}^{19} , $\mathbb{S}^{19}/\mathbb{Z}_m$, or $\mathbb{S}^{19}/S^1 = \mathbb{CP}^9$ with the tensor product representation by $SO(2)SO(10)$ of type C_2 or induced by it. It is easily seen that the Assumption (P) in Lemma 3.4 is satisfied as well. In particular, if $M^K = \mathbb{S}^{19}$, the associated chamber system $\mathcal{C}(M^K)$ is the a building of type C_2 and by Lemma 3.4 we conclude that $\mathcal{C}(M, G)$ is a building.

For the latter two cases, we will use the bundle construction for polar actions to obtain a free S^1 covering of $\mathcal{C}(M; G)$. Guided by our knowledge of the cohomogeneity one diagrams, i.e., data for the cohomogeneity one manifolds $\mathbb{S}^{19}/\mathbb{Z}_m$ or \mathbb{CP}^9 we proceed as follows:

Note that since G_t , G_{ℓ_r} and G_{ℓ_q} are simple groups, only the trivial homomorphism to S^1 exists. Now let \hat{G}_q, \hat{G}_r be the graphs of the projection homomorphisms $G_q \rightarrow S^1$, and $G_r \rightarrow S^1$. We denote the total space of the corresponding principal S^1 bundle over M by P . Then P is a polar $S^1 \cdot G$ manifold, and $\mathcal{C}(P; S^1 \cdot G)$ covers $\mathcal{C}(M; G)$.

Let $\hat{K} \subset \hat{G}_{\ell_t}$ be the graph of K in $S^1 \cdot G$. From 3.5 and our choice of data in $S^1 \cdot G$ it follows that $P^{\hat{K}} \rightarrow M^K$ is the Hopf bundle if $M^K = \mathbb{CP}^9$, and the bundle $S^1 \times_{\mathbb{Z}_m} \mathbb{S}^{19} \rightarrow \mathbb{S}^{19}/\mathbb{Z}_m$ if $M^K = \mathbb{S}^{19}/\mathbb{Z}_m$. In the former case, $\mathcal{C}(P^{\hat{K}})$ is the C_2 building $\mathcal{C}(\mathbb{S}^{19}, SO(2)SO(10))$ and we are done by Lemma 3.4 via 3.5. In the latter case, the action on the reduction $P^{\hat{K}}$ is not primitive, so $\mathcal{C}(P^{\hat{K}})$ is not connected. However, each connected component is the C_2 building $\mathcal{C}(\mathbb{S}^{19}, SO(2)SO(10))$ and hence by 3.5 the corresponding component of $\mathcal{C}(P)$ is a C_3 building covering $\mathcal{C}(M)$. When combined with the previous section, this in turn shows that M^K cannot be a lens space when M is simply connected. \square

PROPOSITION 5.4. *In the Flip (4,4,1) case $\mathcal{C}(M, G)$ or an S^1 covering is a building, with the isotropy representation of $SO(12)/U(6)$ as a linear model.*

Proof. From Lemma 5.1 and Tables 4.4 and 4.3 we obtain the following information about the local data modulo a common \mathbb{Z}_2 kernel: $G_t = Sp(3) \supset Sp(1)^3 = H$, $G_{\ell_q} = Sp(1)Sp(2)$, $G_{\ell_r} = Sp(2)Sp(1)$, $G_q = S^1 Spin(6) Sp(1) \supset \Delta(S^1) \cdot Spin(4) Sp(1) = G_{\ell_t}$, and $G_r = S^1 Spin(5) Sp(1)$.

In this case $G_r \triangleright K = \Delta(S^1) Sp(2) \triangleleft G_{\ell_t} \subset G_q$, and from Lemma 5.2 and Lemma 4.2 we see that the corresponding reduction, M^K is \mathbb{S}^{11} , $\mathbb{S}^{11}/\mathbb{Z}_m$, or $\mathbb{S}^{11}/S^1 = \mathbb{CP}^5$ with the linear tensor product

representation by $\mathrm{SO}(2) \mathrm{SO}(6)$ of type \mathbf{C}_2 or induced by it. It is easily seen that the Assumption (P) in Lemma 3.4 is satisfied as well. In particular, if $M^K = \mathbb{S}^{11}$, the associated chamber system $\mathcal{C}(M^K)$ is the a building of type \mathbf{C}_2 and by Lemma 3.4 we conclude that $\mathcal{C}(M, \mathbf{G})$ is a building.

If $M^K = \mathbb{CP}^5$ or a lens space $\mathbb{S}^{11}/\mathbb{Z}_m$, we proceed as above with an \mathbf{S}^1 bundle construction. Again only the trivial homomorphism to \mathbf{S}^1 exists from \mathbf{G}_t , \mathbf{G}_{ℓ_r} and \mathbf{G}_{ℓ_q} , and we choose $\hat{\mathbf{G}}_q$, $\hat{\mathbf{G}}_r$ to be the graphs of the projection homomorphisms $\mathbf{G}_q \rightarrow \mathbf{S}^1$, and $\mathbf{G}_r \rightarrow \mathbf{S}^1$. We denote the total space of the corresponding principal \mathbf{S}^1 bundle over M by P . As above, P is a polar $\mathbf{S}^1 \cdot \mathbf{G}$ manifold, and $\mathcal{C}(P; \mathbf{S}^1 \cdot \mathbf{G})$ covers $\mathcal{C}(M; \mathbf{G})$.

From 3.5 and our choice of data in $\mathbf{S}^1 \cdot \mathbf{G}$ it follows that $P^K \rightarrow M^K$ is the Hopf bundle if $M^K = \mathbb{CP}^5$, and the bundle $\mathbf{S}^1 \times_{\mathbb{Z}_m} \mathbb{S}^{11} \rightarrow \mathbb{S}^{11}/\mathbb{Z}_m$ if $M^K = \mathbb{S}^{11}/\mathbb{Z}_m$. The proof is completed as above. \square

PROPOSITION 5.5. *In the Flip (2,2,1) case $\mathcal{C}(M, \mathbf{G})$ or an \mathbf{S}^1 covering is a building, with the isotropy representation of $\mathrm{SU}(6)/\mathrm{S}(\mathrm{U}(3) \mathrm{U}(3))$ as a linear model.*

Proof. We begin by verifying our earlier claim (see 2.4) that $\bar{\mathbf{G}}_t$ is connected also in this case. From (5.1) we already know that \mathbf{G}_r and hence $\bar{\mathbf{G}}_r$ is connected, and that its slice representation is the product action of $\bar{\mathbf{G}}_r = \mathrm{SO}(3) \times \mathrm{SO}(2)$ on $\mathbb{R}^3 \oplus \mathbb{R}^2$. The singular isotropy group along \mathbb{R}^2 (away from origin) is $\mathrm{SO}(3)$. Hence, the isotropy group $\bar{\mathbf{G}}_{\ell_q} = \mathrm{SO}(3)$.

On the other hand, suppose $\bar{\mathbf{G}}_t$ is not connected. Then, by 5.1 $\bar{\mathbf{G}}_t = \mathbf{G}_t = \mathrm{PSU}(3) \rtimes \mathbb{Z}_2$ and $\mathbf{G}_{\ell_q} = (\mathrm{S}(\mathrm{U}(2) \mathrm{U}(1))/\mathbb{Z}_3) \rtimes \mathbb{Z}_2$. In particular the slice representation along ℓ_q is by $\bar{\mathbf{G}}_{\ell_q} = \mathrm{PSU}(2) \rtimes \mathbb{Z}_2$ acting on $\mathbb{S}^2 = \mathbb{CP}^1$ where \mathbb{Z}_2 acts by complex conjugation. Contradicting $\bar{\mathbf{G}}_{\ell_q} = \mathrm{SO}(3)$.

The above and Tables 4.4 and 4.3 yield the following information about the local data modulo the \mathbb{Z}_3 kernel: $\mathbf{G}_t = \mathrm{SU}(3) \supset \mathbf{T}^2 = \mathbf{H}$, $\mathbf{G}_{\ell_q} = \mathrm{S}(\mathrm{U}(2) \mathrm{U}(1)) = \mathrm{U}(2)$, $\mathbf{G}_{\ell_r} = \mathrm{S}(\mathrm{U}(1) \mathrm{U}(2)) = \mathrm{U}(2)$, $\mathbf{G}_q = \mathrm{U}(2) \mathrm{U}(2)$. Moreover, $\mathbf{G}_{\ell_t} = \mathbf{T}^3$ and $\mathbf{G}_r = \mathbf{S}^1 \cdot \mathrm{U}(2)$, where the $\mathrm{U}(2)$ factor in \mathbf{G}_r is the face isotropy group of \mathbf{G}_{ℓ_q} .

Here, $\mathbf{G}_r \triangleright \mathbf{K} = \mathbf{T}^2 \triangleleft \mathbf{G}_{\ell_t} \subset \mathbf{G}_q$, and from Lemma 5.2 and Lemma 4.2 we see that the corresponding reduction, M^K is \mathbb{S}^7 , $\mathbb{S}^7/\mathbb{Z}_m$, or $\mathbb{S}^7/\mathbf{S}^1 = \mathbb{CP}^3$ with the linear tensor product representation by $\mathrm{SO}(2) \mathrm{SO}(4)$ of type \mathbf{C}_2 or induced by it. Again, the Assumption (P) in Lemma 3.4 is easily checked to hold. In particular, if $M^K = \mathbb{S}^7$, we conclude as above that $\mathcal{C}(M, \mathbf{G})$ is a building.

For the latter two cases, we are again guided by the reduction for our bundle construction. For $\hat{\mathbf{G}}_t$ we have no choice but $\hat{\mathbf{G}}_t = \{1\} \cdot \mathbf{G}_t$. We let $\hat{\mathbf{G}}_q$ be the graph of the homomorphism $\mathrm{U}(2) \mathrm{U}(2) \rightarrow \mathbf{S}^1$ defined by sending (A, B) to $\det(A) \det(B)^{-1}$, and $\hat{\mathbf{G}}_r$ the graph of the projection homomorphism $\mathbf{G}_r = \mathbf{S}^1 \cdot \mathrm{U}(2) \rightarrow \mathbf{S}^1$. This yields a compatible choice of data for a polar $\mathbf{S}^1 \cdot \mathbf{G}$ action on a principal \mathbf{S}^1 bundle P over M whose corresponding chamber system $\mathcal{C}(P; \mathbf{S}^1 \cdot \mathbf{G})$ is a free \mathbf{S}^1 cover of $\mathcal{C}(M, \mathbf{G})$.

Again from 3.5 and our choice of data in $\mathbf{S}^1 \cdot \mathbf{G}$ it follows that $P^K \rightarrow M^K$ is the Hopf bundle if $M^K = \mathbb{CP}^3$, and the bundle $\mathbf{S}^1 \times_{\mathbb{Z}_m} \mathbb{S}^7 \rightarrow \mathbb{S}^7/\mathbb{Z}_m$ if $M^K = \mathbb{S}^7/\mathbb{Z}_m$, and the proof is completed as above. \square

Remark 5.6. The tensor representation of $\mathrm{SU}(3) \mathrm{SU}(3)$ on $\mathbb{C}^3 \otimes \mathbb{C}^3$ is not polar, but it is polar on the projective space $\mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^3)$. On the other hand, it is necessary in the above construction

of the covering that both G_q and G_r have T^2 factors, since the face isotropy groups $G_{\ell_r} \cong G_{\ell_q} \cong U(2)$ which are subgroups in $G_t = SU(3)$, hence a compatible homomorphism to S^1 will be trivial on the face isotropy groups.

6. NON MINIMAL GRASSMANN SERIES FOR G_q SLICE REPRESENTATION

Recall that there are three infinite families of cases $(1, 1, k), k \geq 1$, $(2, 2, 2k + 1), k \geq 1$ and $(4, 4, 4k + 3), k \geq 0$ corresponding the real, complex and quaternion Grassmann series for the G_q slice representation.

We point out that $(1, 1, 1)$ is special in two ways: There are two scenarios. One of them corresponding to the “Flip” case of $d = 1$ not covered in the previous subsection, the other being “standard”. Yet the standard $(1, 1, 1)$ does not appear as a reduction in any of the general cases $(1, 1, k), k \geq 2$. For the $(2, 2, 3)$ case, there are two scenarios as well, both with *the same local data*!: One of them belonging to the family, the other not. Moreover, each of the cases $(2, 2, 2k + 1)$ with $k \geq 2$ admit a reduction to the “Flip” $(2, 2, 1)$ case, whereas $(2, 2, 3)$ does not.

For the reasons just provided, this subsection will deal with the multiplicity cases $(1, 1, k), k \geq 2$, $(2, 2, 2k + 1), k \geq 2$ and $(4, 4, 4k + 3), k \geq 0$, each of which has a uniform treatment.

Although the case $(2, 2, 3)$ is significantly different from the other general cases to be treated here, we begin by pointing out some common features for all the cases $(1, 1, k), k \geq 2$, $(2, 2, 2k + 1), k \geq 1$ and $(4, 4, 4k + 3), k \geq 0$, i.e., including the case $(2, 2, 3)$.

To describe the information we have about the local data in a uniform fashion, we use $G_d(k)$ to denote $SO(k)$, $SU(k)$ and $Sp(k)$, $k \geq 1$, according to $d = 1, 2$ and $d = 4$, with the exceptional convention that $G_1(-1) = \mathbb{Z}_2$, $G_2(-1) = S^1$ or T^2 , depending on whether the center of K_t is finite or not, and $G_4(0) = G_4(-1) = \{1\}$. Also, we use the symbol “=” to mean “isomorphic” up to a finite connected covering.

LEMMA 6.1. *In all cases G_t is connected as are G_q and G_r when $d \neq 1$. Moreover $K_t = G_d(k)$ with the additional possibility that $K_t = G_d(k) \cdot S^1$ when $d = 2$.*

For the q and r vertex isotropy groups we have: $G_q = G_d(2) G_d(k + 2) \cdot G_d(-1)$, $G_r = G_d(2) G_d(k + 1) \cdot G_d(-1)$. Moreover, the normal subgroup $K \triangleleft G_r$ is $G_d(k + 1) \cdot G_d(-1)$, where $G_d(k + 1)$ is a block subgroup of $G_d(k + 2) \triangleleft G_q$, and if $d = 1$, “ $\cdot G_d(-1)$ ” denotes a nontrivial extension. In particular, $G_q = S(O(2) O(k + 2))$.

Proof. The connectedness claim is a direct consequence of transversality. The proof follows the same strategy in all cases, just simpler when all vertex isotropy groups are connected. The two possibilities for G_t when $d = 2$ correspond to the different rank possibilities for \bar{G}_q , cf. Table 4.4. For these reasons we only provide the proof in the most subtle case of $d = 1$.

First, notice that the effective slice representation $\bar{G}_t = SO(3)$ on T_t^\perp is of type A_2 with principal isotropy group $\bar{H} = \mathbb{Z}_2^2$. Hence, H is an extension of \mathbb{Z}_2^2 by the kernel K_t . On the other hand, $(\bar{G}_q)_0 = SO(2) SO(k + 2)$ (cf. Table 4.4), and $\bar{G}_q \subset O(2) O(k + 2)$, up to a possible quotient by a diagonal \mathbb{Z}_2 in the center if k is even. Therefore, H is also an extension of $SO(k)$, $SO(k) \cdot \mathbb{Z}_2$ or $SO(k) \cdot \mathbb{Z}_2^2$ by K_q . This together with Lemma 2.3, implies that $K_t = SO(k)$ and hence $G_t = SO(3) SO(k)$. In particular, $H = SO(k) \times \mathbb{Z}_2^2$.

We conclude that $G_{\ell_r} = O(2) SO(k)$, and similarly, $G_{\ell_q} = O(2) SO(k)$, acting on the normal sphere S^1 with principal isotropy group H . Thus $K_{\ell_r} = SO(k) \times \mathbb{Z}_2 = K_t \times \mathbb{Z}_2$. Since $K_q \triangleleft K_{\ell_r}$,

we get easily that $K_q = \{1\}$ or \mathbb{Z}_2 , since $K_q \cap K_t = \{1\}$. On the other hand, as a subgroup of G_q , $G_{\ell_r} = \Delta(\mathrm{O}(2)) \mathrm{SO}(k)$. Hence G_q contains exactly two connected components, whose identity component is $\mathrm{SO}(2) \mathrm{SO}(k+2) \supset (G_{\ell_r})_0$. All in all it follows that, $G_q = \mathrm{S}(\mathrm{O}(2) \mathrm{O}(k+2))$. The rest of the proof is straightforward. \square

Note that K_t contains $G_d(k)$ as a normal subgroup. The fact that the reduction $M^{G_d(k)}$ with the action by the identity component of its normalizer, $N_0(G_d(k))$, will give a geometry of type A_3 or C_3 will play an important role in the $d = 1, 2$ cases below (cf. 3.6).

In what follows we will consider the reduction $M^{K'}$ by $K' = G_d(k+1) \triangleleft G_d(k+1) \cdot G_d(1) = K \subset G_r$ rather than the one by K .

LEMMA 6.2. *The cohomogeneity one $N(K')$ manifold $M^{K'}$ has multiplicity pair $(d, 2d-1)$, and the action is not equivalent to the reducible cohomogeneity one action on $\mathbb{S}^{2d-1} * \mathbb{S}^d$.*

Proof. For simplicity we give a proof for $d = 2$, all other cases are the same.

First note that the orbit space of the cohomogeneity one $N(K')$ -action is \overline{rq} , and the two singular isotropy groups (mod kernel) are $\mathrm{SU}(2) \cdot \mathbb{S}^1$ and $\mathrm{SU}(2) \cdot \mathbb{T}^2$ respectively, with principal isotropy group \mathbb{T}^2 . Hence the multiplicity pair is $(2, 3)$.

To prove that it is not reducible, we argue by contradiction. Indeed, if $M^{K'}$ is equivariantly diffeomorphic to $\mathbb{S}^2 * \mathbb{S}^3$ with the product action of $\mathrm{SU}(2) \mathrm{U}(2)$, it follows that the normal subgroup $\mathrm{SU}(2) \triangleleft G_q$ is also normal in $N(K')$. By primitivity $G = \langle G_r, G_q \rangle = \langle N, G_q \rangle$ and hence $\mathrm{SU}(2)$ is normal in G . On the other hand, the face isotropy group $G_{\ell_r} \subset G_r$ contains a subgroup $\mathrm{SU}(2)$ which sits as $\Delta(\mathrm{SU}(2)) \subset G_q$. Therefore, the projection homomorphism $p : G \rightarrow \mathrm{SU}(2)$ is an epimorphism on $\Delta(\mathrm{SU}(2))$. However, since it sits in $\mathrm{SU}(3) \triangleleft G_t$ it must be trivial, because any homomorphism from $\mathrm{SU}(3)$ to $\mathrm{SU}(2)$ is trivial. A contradiction. \square

When $d = 1$ this is not immediately of much help since there are several positively curved irreducible cohomogeneity one manifolds with multiplicity pair $(1, 1)$ (cf. Tables A and E in [GWZ]) whose associated chamber system is not of type C_2 . However, when $d = 2$, respectively $d = 4$ corresponding to multiplicity pairs $(2, 3)$, respectively $(4, 7)$ we read off from the classification in [GWZ] that

COROLLARY 6.3. *The universal covering of $M^{K'}$ is equivariantly diffeomorphic to a linear action of type C_2 on \mathbb{S}^{11} , \mathbb{CP}^5 or \mathbb{HP}^2 when $d = 2$, and on \mathbb{S}^{23} when $d = 4$.*

We are now ready to deal with each family individually, beginning with $d = 1$, i.e. with the standard $(1, 1, k \geq 2)$ case, where the (almost) effective slice representation at $q \in Q$ is the defining tensor product representation of $\mathrm{SO}(2) \mathrm{SO}(k+2)$.

PROPOSITION 6.4. *In the standard $(1, 1, k)$ case with $k \geq 2$, the associated chamber system $\mathcal{C}(M; G)$ is a building, with the isotropy representation of $\mathrm{SO}(k+3)/\mathrm{SO}(3) \mathrm{SO}(k)$ as a linear model.*

Proof. By Lemma 6.1 $K_t = \mathrm{SO}(k)$, which is a normal subgroup of the principal isotropy group H . Consider the reduction M^{K_t} with the action of its normalizer $N(K_t)$, once again a polar action with the same section Σ . By Lemma 6.1 it is clear that the identity component of $N(K_t) \cap G_q$ is \mathbb{T}^2 . Hence, the subaction by $N_0(K_t)$, the identity component of $N(K_t)$, is of type A_3 , with a

right angle at q . Therefore, from the classification of A_3 geometries (cf. section 7 in [FGT]) it is immediate that, the universal cover of M^{K_i} is equivariantly diffeomorphic to \mathbb{S}^8 with the linear action of $\mathrm{SO}(3)\mathrm{SO}(3)$. In particular, if the section $\Sigma = \mathbb{S}^2$, then $M^{K_i} = \mathbb{S}^8$ and the chamber complex for the subaction is a building of type A_3 , and we are done by Remark 3.6, since Property (P) is clearly satisfied for $K = \mathrm{SO}(k+1) \triangleleft G_r$.

It remains to prove that M^{K_i} is simply connected. Consider the normal subgroup $\mathrm{SO}(2) \triangleleft G_q$, and the fixed point component $M^{\mathrm{SO}(2)}$, a homogeneous manifold of positive curvature with dimension at least two, since $M^{\mathrm{SO}(2)} \cap M^{K_i} \subset M^{K_i}$ is of dimension 2. Since the identity component of the isotropy group, $(G_q)_0 = \mathrm{SO}(2)\mathrm{SO}(k+2)$, we see that $M^{\mathrm{SO}(2)} = \mathbb{S}^{k+2}$ or \mathbb{RP}^{k+2} , according to $M^{K_i} \cap M^{\mathrm{SO}(2)} = \mathbb{S}^2$ or \mathbb{RP}^2 , equivalently, according to $M^{K_i} = \mathbb{S}^8$ or \mathbb{RP}^8 . We argue by contradiction. If $M^{\mathrm{SO}(2)} = \mathbb{RP}^{k+2}$, then the identity connected component of the normalizer $N(\mathrm{SO}(2))$ acts transitively on it with principal isotropy group $\mathrm{SO}(2)\mathrm{O}(k+2) \subset G_q$. Hence $G_q = \mathrm{SO}(2)\mathrm{O}(k+2)$, a contradiction, since $G_q = \mathrm{S}(\mathrm{O}(2)\mathrm{O}(k+2))$. \square

PROPOSITION 6.5. *In the standard $(2, 2, 2k+1)$ case, with $k \geq 2$ the chamber system $\mathcal{C}(M; G)$ is covered by a building, with the isotropy representation of $\mathrm{U}(k+3)/\mathrm{U}(k)\mathrm{U}(3)$ as a linear model.*

Proof. First note that the reduction $M^{\mathrm{SU}(k)}$, where $\mathrm{SU}(k) \triangleleft K_i$, $k \geq 2$, is a positively curved cohomogeneity two manifold of type C_3 with multiplicity triple $(2, 2, 1)$. Moreover, $\mathrm{SU}(k)$ is a block subgroup in $K' \subset K$, where $K' = \mathrm{SU}(k+1) \subset \mathrm{SU}(k+2) \triangleleft G_q$ and of course $M^{K'} \subset M^{\mathrm{SU}(k)}$.

We will prove that both reductions above are simply connected, by appealing to the Connectivity Lemma 2.6 of Wilking [Wi3]. To do this we now proceed to prove that $\mathrm{codim} M^{K'} \subset M^{\mathrm{SU}(k)} = 6$, and $\mathrm{codim} M^{\mathrm{SU}(k)} \subset M = 6k$.

By the Spherical isotropy Lemma 2.8, every irreducible isotropy subrepresentation of $K' = \mathrm{SU}(k+1)$ is the defining representation μ_{k+1} . From Table B in [GWZ] and the above fact that $\mathrm{SU}(k+2) \supset K'$ it follows that, there is a simple normal subgroup $L \triangleleft G$ such that, $\mathrm{SU}(k+2) \triangleleft G_q$ projects to a block subgroup of L where $L = \mathrm{SU}(n)$ if $k \geq 4$, $L = \mathrm{SU}(n)$ or $\mathrm{SO}(n)$ if $k = 3$, and finally $L = \mathrm{SU}(n)$, $\mathrm{SO}(n)$ or one of the exceptional Lie groups $F_4 \subset E_6 \subset E_7 \subset E_8$, if $k = 2$.

On the other hand, by the Flip Proposition 5.5 the normalizer $N(\mathrm{SU}(k))$ is either $\mathrm{SU}(3)\mathrm{SU}(3)$ or $\mathrm{U}(3)\mathrm{SU}(3)$ modulo K_i . Since $\mathrm{SU}(k) \triangleleft K_i$ is a block subgroup in K' , this together with the above implies that in fact $L = \mathrm{SU}(k+3)$ for all $k \geq 2$, and only one such factor exist. In particular, the K' -isotropy representation along ℓ_i contains exactly 3 copies of μ_{k+1} , one copy along the normal slice $T_{\ell_i}^\perp$, and two copies along the orbit G/G_{ℓ_i} . Therefore, the codimension of $M^{K'}$ in M is $6(k+1)$, and hence the codimension of $M^{\mathrm{SU}(k)}$ in M is $6k$. By the Connectivity lemma 2.6 of Wilking, we conclude that $\pi_i(M) \cong \pi_i(M^{\mathrm{SU}(k)})$ for $i \leq 2$, by induction on k . In particular, $M^{\mathrm{SU}(k)}$ is simply connected and hence \mathbb{S}^{17} if $\dim(M)$ is odd and \mathbb{CP}^8 if $\dim(M)$ is even, by the Flip Proposition 5.5. Since Assumption (P) in Lemma 3.4 is satisfied we conclude from 3.6 that $\mathcal{C}(M; G)$ is a building if $\dim(M)$ is odd.

It remains to prove that $\mathcal{C}(M; G)$ is covered by a building if $\dim(M)$ is even. In this case, by the above we know that $\pi_2(M) \cong \pi_2(M^{\mathrm{SU}(k)}) \cong \mathbb{Z}$. On the other hand, from the Transversality Lemma 2.5 it follows that $\pi_2(M) \cong \pi_2(G/G_i)$, and hence G_i contains at least an S^1 in its center, i.e., $\mathrm{SU}(3)\mathrm{U}(k) \triangleleft G_i$. By Lemma 6.1 we get that, both G_q and G_r have at least a T^2 factor, and we are now in the same situation as in the proof of Lemma 5.5 above. As a consequence we

can proceed with the same construction of a principal S^1 bundle P over M and conclude that its associated chamber system is a building covering $\mathcal{C}(M; G)$. \square

PROPOSITION 6.6. *In the standard $(4, 4, 4k+3)$ case where $k \geq 0$, the chamber system $\mathcal{C}(M; G)$ is a building, with the isotropy representation of $Sp(k+3)/Sp(k)Sp(3)$ as a linear model.*

Proof. Since the Assumption (P) for $K' = Sp(k+1)$ in Lemma 3.4 is easily seen to be satisfied, it suffices by Corollary 6.3 to prove that $M^{K'}$ is simply connected. As in the proof of the general $(2, 2, 2k+1)$ case above this is achieved via Wilkings Connectivity Lemma 2.6.

Consider the normal subgroup $Sp(2) \triangleleft G_q$. It is clear that $M^{Sp(2)}$ is a homogeneous space with a transitive action by the identity component of its normalizer $N_0(Sp(2))$ with isotropy group G_q . By the classification of positively curved homogeneous spaces we get that $M^{Sp(2)}$ is either $S^{4(k+3)-1}$ or $RP^{4(k+3)-1}$. Moreover, the universal cover $\tilde{N}_0(Sp(2))$ is $Sp(k+3)Sp(2)Sp(1)$, and in particular has the same rank as G by the Rank Lemma.

On the other hand, by Lemma 2.8 and Table B in [GWZ] it follows that, G contains a normal subgroup isomorphic to $Sp(n)$ so that $K' \subset Sp(k+2) \subset Sp(k+3) \subset Sp(n)$ is in a chain of block subgroups. Up to a finite cover, we let $G = Sp(n) \cdot L$. On the other hand, by Corollary 6.3 we know that $N_0(K') = Sp(2)Sp(3)K'$. This together with the information on $\tilde{N}(Sp(2))$ implies that $G = Sp(k+3) \cdot L$. As in the proof of the $(2, 2, 2k+1)$ case we see that the isotropy representation of K' , along ℓ_i contains exactly three copies of ν_{k+1} , one copy along the normal slice $T_{\ell_i}^\perp$, and two copies along the orbit G/G_{ℓ_i} . In particular, the codimension of $M^{K'}$ in M is $12(k+1)$. Recalling that the dimension of $M^{K'}$ is 23, it follows again by connectivity and induction on k as before that $M^{K'}$ is simply connected. \square

7. MINIMAL GRASSMANN G_q SLICE REPRESENTATION

This section will deal with the multiplicity cases $(1, 1, 1)$ and $(2, 2, 3)$, including the appearance of an exceptional Cayley plane action. In all previous cases all reductions considered have been irreducible polar actions. Here, however, we will encounter reductions, that are *reducible cohomogeneity two actions*, and we will rely on the independent classification of such actions in sections 6 and 7 of [FGT].

We begin with the $d = 2$ case, where by 6.3 we know that the universal covering $\tilde{M}^{K'}$ of the reduction $M^{K'}$ is diffeomorphic to S^{11} , CP^5 or HP^2 . The first two scenarios follow the outline of the general $(2, 2, 2k+1)$ case, whereas the latter is significantly different.

PROPOSITION 7.1. *In the case of multiplicities $(2, 2, 3)$, $\mathcal{C}(M; G)$ is covered by a building, with the isotropy representation of $U(7)/U(4)U(3)$ as a linear model, provided $M^{K'}$ is not diffeomorphic to HP^2 .*

Proof. By Lemma 6.1, G_i is either $SU(3)$ or $U(3)$ depending on whether K_i is finite or S^1 . In the latter case, the reduction M^{K_i} is a positively curved cohomogeneity two manifold of type C_3 with multiplicity triple $(2, 2, 1)$, as in the general $(2, 2, 2k+1)$ case, where $k \geq 2$ (cf. 6.5). Therefore, $N_0(K_i)/K_i = SU(3) \cdot SU(3)$ or $U(3) \cdot SU(3)$, by the Flip Proposition 5.5. The desired result follows, as in the proof of Proposition 6.5.

From now on we assume that, up to finite kernel, $G_t = \mathrm{SU}(3)$, and correspondingly, $G_q = \mathrm{U}(2) \mathrm{SU}(3)$, and $G_r = \mathrm{U}(2) \mathrm{SU}(2)$. Moreover, $K' = \mathrm{SU}(2)$, and from our assumption on the reduction $M^{K'}$, by Corollary 6.3 the normalizer $N(K')$ contains $\mathrm{SU}(2) \mathrm{SU}(2) \mathrm{SU}(3)$ as its semisimple part. On the other hand, by the Rank Lemma 2.7 we know that $\mathrm{rk}(G) = 5$ (resp. $\mathrm{rk}(G) = 4$) if $\dim(M)$ is odd (resp. even). In particular, $\mathrm{SU}(2) \mathrm{SU}(2) \mathrm{SU}(3)$ is a maximal rank subgroup of G if $\mathrm{rk}(G) = 4$. In this case, it is immediate, by Borel and de Siebenthal [BS] (see the Table on page 219), that G is not a simple group of rank 4. Similarly, we claim that G is not a simple group when its rank is 5: Indeed if so, by Lemma 2.8 and Table B in [GWZ], it would follow that $G = \mathrm{SU}(6)$ and $K' = \mathrm{SU}(2) \subset \mathrm{SU}(3) \triangleleft G_q$ is a block subgroup. This, however, is not possible, since then $N(K')$ would contain $\mathrm{SU}(4)$. Thus, $G = L_1 \cdot L_2$, where L_1, L_2 are nontrivial Lie groups. Without loss of generality, we assume that the projection of $\mathrm{SU}(3) \triangleleft G_q$ to L_2 has nontrivial image. But then $\mathrm{SU}(3)$ must be contained in L_2 , because otherwise, the normalizer $N(K')$ would be much smaller than $\mathrm{SU}(2) \mathrm{SU}(2) \mathrm{SU}(3)$. By Primitivity 2.2 it is easy to see that G_t is diagonally imbedded in $L_1 \cdot L_2$, since $G = \langle G_t, G_{\ell_t} \rangle = \langle G_t, K' \rangle$. In particular, both L_1 and L_2 have rank at least two since the projections from G_t are almost imbeddings, i.e., have finite kernel. If both L_1 and L_2 have rank two, it is easy to see that, $L_1 = \mathrm{SU}(3)$ and $K' \subset L_2$, where $L_2 = \mathrm{SU}(3)$ or G_2 . Neither scenario is possible: For the latter since, by the primitivity, $G = \langle \Delta(\mathrm{SU}(3)), K' \rangle = \mathrm{SU}(3) \cdot \mathrm{SU}(3)$, while for the former the semisimple part of $N(K')$ is L_1 . Therefore $\mathrm{rk}(G) = 5$ and once again by Lemma 2.8 and Table B in [GWZ], $G = \mathrm{SU}(3) \mathrm{SU}(4)$.

Note that $\dim M = 21$ and the principal orbit of K' in M is of dimension at least 2. In particular, it follows from Wilkings Connectivity Lemma 2.6 that $M^{K'}$ is simply connected. Thus, as in the general case the desired result follows from Lemma 3.4. \square

PROPOSITION 7.2. *In the case of multiplicities $(2, 2, 3)$, M is equivariantly diffeomorphic to the Cayley plane $\mathbb{O}\mathbb{P}^2$ with an isometric polar action by $\mathrm{SU}(3) \cdot \mathrm{SU}(3)$, provided $M^{K'}$ is diffeomorphic to $\mathbb{H}\mathbb{P}^2$.*

Proof. Recall that $K' = \mathrm{SU}(2) \triangleleft G_{\ell_t}$. By Lemma 2.8 and the slice representation of G_{ℓ_t} it follows that, every irreducible subrepresentation of K' on the normal space to $M^{K'}$ is the standard representation μ_2 on \mathbb{C}^2 . In particular, the codimension of $M^{K'}$ is a multiple of 4, and so M has dimension divisible by 4. By 6.1 the isotropy group $G_t = \mathrm{SU}(3)$ or $\mathrm{U}(3)$, and correspondingly, $G_q = \mathrm{U}(2) \mathrm{SU}(3)$ or $\mathrm{U}(2) \mathrm{U}(3)$, and $G_r = \mathrm{U}(2) \mathrm{SU}(2)$ or $\mathrm{U}(2) \mathrm{U}(2)$. By the Rank Lemma $\mathrm{rk}(G) = \mathrm{rk}(G_q) = 4$ or 5.

By Lemma 2.8 the isotropy representations of $K' \subset \mathrm{SU}(3) \triangleleft G_q$, as well as of $\mathrm{SU}(2) \subset G_{\ell_t} \subset G_t$, are spherical transitive. By Table B in [GWZ] it follows that, G can not be a simple group of rank 5, and moreover, G can not contain F_4 , $\mathrm{Sp}(4)$, $\mathrm{SO}(8)$ and $\mathrm{SO}(9)$ as a normal subgroup, since if so, the semisimple part of $N_0(K')/K'$ would not be $\mathrm{SU}(3)$, a contradiction to our assumption on the reduction $M^{K'}$, for which $N_0(K')/K' = \mathrm{SU}(3) \cdot S^1$. On the other hand, note that the identity component of the normalizer $N_0(G_t) = G_t$ since G_t is a maximal isotropy group and hence $N_0(G_t)/G_t$ acts freely on the positively curved fixed point set M^{G_t} of even dimension. Therefore, G can not contain $\mathrm{SU}(5)$ as a normal subgroup, since otherwise, G_t would be a block subgroup in $\mathrm{SU}(5)$ and hence $N_0(G_t)/G_t$ would not be trivial. Consequently, G is not a simple group, and moreover, $G = L_1 \cdot L_2$, where $\mathrm{SU}(3) \triangleleft G_t$ is diagonally imbedded in G . In particular, both L_1 and L_2 contain $\mathrm{SU}(3)$ as subgroups. It is easy to see that, $\mathrm{SU}(3) \triangleleft G_q \subset G = L_1 \cdot L_2$ is a subgroup in either L_1 or L_2 , say in L_2 . Hence, $K' \subset L_2$, and $L_1 \triangleleft N(K')$. It follows that $L_1 =$

$SU(3)$. Furthermore, L_2 can neither be a semi-simple group of rank 3 or G_2 , since otherwise, $N_0(K')/K'$ contains a rank 3 semisimple group. Hence, L_2 is $SU(3)$ or $U(3)$. The latter, however, is impossible: Indeed, in this case $G_t = U(3)$, and the center $S^1 \subset Z(G)$ would be contained in K_t , and hence in every principal isotropy groups (the center is invariant under conjugation) thus $M^{S^1} = M$.

In summary we have proved that $G = SU(3) \cdot SU(3)$ (indeed a quotient group by $\Delta(\mathbb{Z}_3)$), with $G_t = SU(3)$ diagonally imbedded in G . We claim that this combined with the above analysis of the isotropy groups modulo conjugation will force the polar data $(G_t, G_q, G_r) \subset G$ (noting that face isotropy groups are intersections of vertex isotropy groups) to be $(G_t, G_q, G_r) = (\Delta(SU(3)), U(2) \cdot SU(3), S(U(2) U(2)))$, where $U(2) \subset SU(3)$ is the upper 2×2 block subgroup in $SU(3)$, and $S(U(2) U(2)) \subset SU(3) \cdot SU(3)$ is the product of the lower 2×2 block subgroups. In other words, by the recognition theorem for polar actions [GZ] there is at most one such polar action. - On the other hand the unique action by the maximal subgroup $SU(3) \cdot SU(3) \subset F_4$, the isometry group of the Cayley plane $\mathbb{O}P^2$ is indeed polar of type C_3 [PTh].

To prove the above claim, by conjugation we may assume that $G_t = \Delta(SU(3))$ and $G_q = U(2) \cdot SU(3)$ as claimed. Moreover, up to conjugation by an element of the face isotropy group $G_{\ell_r} = G_t \cap G_q$, we may further assume that $K' \triangleleft G_{\ell_r} \subset G_q$ is the lower 2×2 block subgroup in the second factor $SU(3) \triangleleft G$. Note that K' is a normal subgroup of G_r , indeed the second factor of $SU(2) \cdot SU(2) \triangleleft G_r \subset SU(3) \cdot SU(3)$. Since $G_{\ell_q} = \Delta(SU(3)) \cap G_r$, it follows that $SU(2) \cdot SU(2) \triangleleft G_r$ is the product of the lower 2×2 block subgroups. Since $G_r = \langle SU(2) \cdot SU(2), H \rangle$ where $H = \Delta(T^2)$ is the principal isotropy group, the desired assertion follows. \square

Next we deal with the case of multiplicity $(1, 1, 1)$, where there are two scenarios: One is naturally viewed as part of the infinite family $(1, 1, k)$, whereas the other should be viewed as the flip case with $d = 1$.

We point out that unlike all other cases an S^3 chamber system cover arises in the first case, corresponding to a polar action of $SO(3) SO(3)$ on $\mathbb{H}P^2$.

PROPOSITION 7.3. *For the multiplicity $(1, 1, 1)$ case, the chamber system $\mathcal{C}(M; G)$ is covered by a building, with the isotropy representation of either $SO(7)/SO(4) SO(3)$, or of $Sp(3)/U(3)$ as a linear model.*

Proof. Recall that, $\bar{G}_t = SO(3)$, and $\bar{H} = \mathbb{Z}_2^2$. We first claim that the identity component $(G_t)_0 = SO(3)$. To see this, recall that the kernel $K_t \subset K_{\ell_r}$, and \bar{G}_q is either $SO(2) SO(3)$ or $S(O(2) O(3))$. The claim follows since, if $\dim K_t \geq 1$ or $(G_t)_0 = S^3$, then $K_t \cap K_q$ is nontrivial, a contradiction to Lemma 2.3. From this we also conclude that $(G_q)_0$ is not $S^1 \times S^3$, since otherwise again $K_t \cap K_q$ is non-trivial. Hence it is isomorphic to either $SO(2) SO(3)$ (the “standard” case) or to the 2 fold covering $U(2)$ of $SO(2) SO(3)$ (the “flip” case). By the Rank Lemma 2.7 it follows that $\text{rank } G \leq 3$.

We start with the following observation:

- Let z be cyclic subgroup of the principal isotropy group H with non-trivial image $[z] \subset \bar{H}$. Then the action by $N_0(z)$ on the reduction M^z is a *reducible* polar action of cohomogeneity 2. To see this note that the type t orbit in the reduction is no longer a vertex. Indeed the normalizer of $[z] \subset \bar{H} \subset SO(3)$ is $O(2)$.

In addition, note that the identity component of every face isotropy group is S^1 . By the Dual Generation Lemma 7.2 in [FGT] we conclude that

- 7.3.1. *The semisimple part of $N_0(z)$ has rank at most one.*

To proceed we will prove that

- (a). G is not a simple group of rank 3.

This is a direct consequence of 7.3.1 combined with the following algebraic fact: If G is a rank 3 simple group, i.e., one of $SO(6) = SU(4)$, $SO(7)$ or $Sp(3)$ (up to center), then, the normalizer of any order 2 subgroup $\mathbb{Z}_2 \subset SO(3) = (G_t)_0$ contains a semisimple subgroup of rank at least 2. The algebraic fact is easily established by noticing that the inclusion map $SO(3) \rightarrow G$ either can be lifted to a homomorphism into one of the four matrix groups, or $SO(3)$ sits in the quotient image of a diagonally imbedded $SU(2)$ in one of the matrix groups.

Next we are going to prove that

- (b). If G is a rank 2 group, then either $(M, G) = (\mathbb{CP}^5, SU(3))$ or $(\mathbb{HP}^2, SO(3) SO(3))$ up to equivariant diffeomorphism.

Exactly as in Case (a), we can exclude G being $SO(5)$ since a subgroup $\mathbb{Z}_2 \subset \bar{H} \subset (G_q)_0$ will have a normalizer containing $SO(4)$. We now exclude G being the exceptional group G_2 . Otherwise, $(G_q)_0$ must be $U(2)$, and contained in either an $SO(4) \subset G_2$ or an $SU(3) \subset G_2$ by Borel-Siebenthal [BS]. The center $\mathbb{Z}_2 \subset U(2)$ is in K_q . For the same reason as above, $U(2)$ is not in $SO(4) \subset G_2$. Finally, if $U(2) \subset SU(3) \subset G_2$, the q orbit $(Gq)^{\mathbb{Z}_2}$ in the reduction $M^{\mathbb{Z}_2}$ contains $(G_2 / SU(3))^{\mathbb{Z}_2} = (\mathbb{S}^6)^{\mathbb{Z}_2} = \mathbb{S}^2$. Again by the Dual Generation Lemma 7.2 of [FGT] this is impossible, since the identity component of the isotropy group of the face opposite of q is a circle, which cannot act transitively on the orbit $(Gq)^{\mathbb{Z}_2}$. Therefore, up to local isomorphism, G is $SO(3) SO(3)$ or $SU(3)$ respectively. One checks that the corresponding isotropy group data are given by $G_t = \Delta(SO(3)) \subset SO(3) SO(3)$, and $G_q = O(2) SO(3) \subset SO(3) SO(3)$, respectively by $G_t = SO(3) \subset SU(3)$ (inclusion induced by the field homomorphism), and $G_q = U(2) \subset SU(3)$ as a block subgroup. The recognition theorem then yields (b).

- (c). Now suppose $G = L_1 \cdot L_2$, where L_i is a rank i Lie group.

If L_1 acts freely on M , then $L_1 = S^1$, $SO(3)$, or S^3 , and L_2 acts on M/L_1 in a polar fashion of type C_3 . Hence, M/L_1 is even dimensional and thus \mathbb{CP}^5 or \mathbb{HP}^2 by (b). In either case, we know that the universal cover $\tilde{\mathcal{C}}$ of the chamber system $\mathcal{C}(M/L_1, L_2)$ is a building. Since $\mathcal{C}(M, G)$ is a connected chamber system covering $\mathcal{C}(M/L_1, L_2)$ it follows that $\tilde{\mathcal{C}}$ is the universal cover of $\mathcal{C}(M, G)$.

Now consider the remaining case where

- L_1 does not act freely on M , and we let $\mathbb{Z}_m \subset L_1$ be a cyclic group such that $M^{\mathbb{Z}_m} \neq \emptyset$.

Note that G can not be $SO(3) \cdot T^2$, since, then G_t and $SO(3) \triangleleft G_q$ would be the same simple group factor, which is absurd. In particular, the semi-simple part of G has rank at least two. Thus from now on we may assume that L_2 is a rank two semi-simple group. Moreover, by the argument in Case (b) it is immediate that in fact L_2 is either $SO(4)$ or $SU(3)$.

Notice that:

- If K_t is not trivial, then $(M^{K_t}, N_0(K_t))$ is a polar manifold with the same section, which is of type A_3 . By the Connectivity Lemma 2.6 it follows that M^{K_t} is simply connected. Hence, from the classification of A_3 geometries, M^{K_t} is diffeomorphic to \mathbb{S}^8 , and the chamber system of $(M^{K_t}, N_0(K_t))$ is a building. By 3.6 $\mathcal{C}(M, G)$ is a building.

Therefore, we may assume in the following that $K_t = \{1\}$, hence $G_t = \mathrm{SO}(3)$. It follows that, G_q is either $\mathrm{S}(\mathrm{O}(2) \mathrm{O}(3))$ or $\mathrm{U}(2)$.

We split the rest of the proof according to L_1 abelian or not. In either case note that the normalizer $N_0(\mathbb{Z}_m)$ is $\mathrm{S}^1 \cdot L_2$. From this we get immediately that $\mathbb{Z}_m \not\subset H$, by appealing to 7.3.1.

(ci) $G = \mathrm{S}^1 \cdot L_2$.

It suffices to prove that the S^1 action is free, since then the situation reduces to the previous rank 2 case.

Note that \mathbb{Z}_m is normal in G . From this and the above it follows that \mathbb{Z}_m is neither in G_t nor in G_{ℓ_t} . To see this, if $\mathbb{Z}_m \subset G_{\ell_t}$ then $(\bar{G}_q)_{\ell_t} \subset \bar{G}_q$ would contain a non-trivial normal subgroup of \bar{G}_q contradicting Table 4.4. The proof in the other case is similar but simpler. Hence, $M^{\mathbb{Z}_m}$ is either the orbit $G \cdot r$ or $G \cdot q$.

Assuming $M^{\mathbb{Z}_m} = G / G_r$, it is immediate that $L_2 = \mathrm{SU}(3)$ from the list of positively curved homogeneous spaces. On the other hand, notice that G_r is not connected, indeed $(G_r)_0 = \mathrm{T}^2$ and $G_r \supset G_{\ell_q} \supset \mathrm{O}(2)$, it follows that G / G_r is not simply connected. However, G / G_r is a totally geodesic submanifold in M which has dimension 11. A contradiction to Wilking's Connectivity Lemma 2.6.

Assuming $M^{\mathbb{Z}_m} = G / G_q$, corresponding to $L_2 = \mathrm{SU}(3)$ or $\mathrm{SO}(4)$, the universal cover of G / G_q is a sphere of dimension either 5 or 3. The latter case is ruled out as follows: If $G_q = \mathrm{U}(2)$ then $K_q = \mathbb{Z}_2$ is in the center of $\mathrm{U}(2)$ hence also in the center of G . This is impossible, since $K_q \subset H$ and G acts effectively on M by assumption. If $G_q = \mathrm{S}(\mathrm{O}(2) \mathrm{O}(3))$ there are no non-trivial homomorphisms to S^1 , hence $G_q \subset \mathrm{SO}(4)$, which is impossible. For the former case, $G_q = \mathrm{U}(2)$ and $G = \mathrm{U}(3)$, with action on G / G_q equivalent to the standard linear action on a 5-dimensional spherical space form with \mathbb{Z}_m in the kernel. Thus, $G_q \supset \mathbb{Z}_m \times \mathrm{U}(2)$, a contradiction.

(cii) $G = L_1 \cdot L_2$, where L_1 is a simple rank one group, i.e., either S^3 or $\mathrm{SO}(3)$.

We will show that in this case $G = \mathrm{SO}(3) \mathrm{SO}(4)$, with local data $G_q = \mathrm{S}(\mathrm{O}(2) \mathrm{O}(3)) \subset G$, and $G_t = \Delta(\mathrm{SO}(3)) \subset G$ forcing all data to coincide with those of the isotropy representation of $\mathrm{SO}(7) / \mathrm{SO}(3) \mathrm{SO}(4)$, and hence M with the action of G is determined via recognition.

We first prove that $L_2 = \mathrm{SO}(4)$. If not, we start with an observation that, $L_1 = \mathrm{SO}(3)$, and moreover, G_t is a diagonally imbedded subgroup in $L_1 \cdot L_2$. Indeed, otherwise, an order 2 element $z \in H \subset G_t$ will have a normalizer $N_0(z)$ which contains a rank 2 semisimple subgroup, contradicting 7.3.1. For the same reason, as above, we see that $G_q \neq \mathrm{U}(2)$ and hence, $G_q = \mathrm{S}(\mathrm{O}(2) \mathrm{O}(3))$. Similarly by 7.3.1, $\mathrm{SO}(3) \triangleleft G_q$ must be diagonally embedded in $L_1 \cdot L_2$. This is impossible since then $N(\mathrm{SO}(3)) / \mathrm{SO}(3)$ is finite, but $(G_q)_0 \subset N(\mathrm{SO}(3))$.

Finally, given that $L_2 = \mathrm{SO}(4)$ it follows as above that $G_q \neq \mathrm{U}(2)$, hence $G_q = \mathrm{S}(\mathrm{O}(2) \mathrm{O}(3))$. Since $G_t = \mathrm{SO}(3)$ and $G_{\ell_r} = \mathrm{O}(2)$ sits diagonally in G_q it follows that G_t sits diagonally in $L_1 \cdot L_2$, in particular $L_1 = \mathrm{SO}(3)$. Using the same arguments as above we see that $\mathrm{SO}(3) \triangleleft G_q$ is in L_2 . All together, all isotropy data are determined. \square

8. EXCEPTIONAL G_q SLICE REPRESENTATION

This section will deal with the remaining cases, all of which are exceptional with multiplicities $(1, 1, 5)$, $(2, 2, 2)$, $(4, 4, 5)$, and $(1, 1, 6)$. All but the latter will occur, and the case of $(1, 1, 5)$ will include an exceptional action on the Cayley plane.

PROPOSITION 8.1. *In the case of the multiplicities $(1, 1, 5)$ where the (effective) slice representation at T_q^\perp is the tensor representation of $\mathrm{SO}(2) \cdot \mathrm{G}_2$ on $\mathbb{R}^2 \otimes \mathbb{R}^7$, either M is equivariantly diffeomorphic to the Cayley plane $\mathbb{O}\mathbb{P}^2$ with an isometric polar action by $\mathrm{SO}(3) \cdot \mathrm{G}_2$ or $\mathcal{C}(M, \mathrm{G})$ is a building, with the tensor product representation of $\mathrm{SO}(3) \cdot \mathrm{Spin}(7)$ on $\mathbb{R}^3 \otimes \mathbb{R}^8$ as a linear model.*

Proof. By the Transversality Lemma 2.5 we conclude that G_t is connected since $\mathrm{G} / \mathrm{G}_t$ is simply connected. The kernel K_t is a normal subgroup in G_t , as well as of the principal isotropy group H with quotients $\mathrm{G}_t / \mathrm{K}_t = \mathrm{SO}(3)$, and $\mathrm{H} / \mathrm{K}_t = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ respectively (cf. Table 4.3). By the Slice Lemma 2.3 K_t acts effectively on the q -slice. Combining this with Table 4.4 where $(\tilde{\mathrm{G}}_q)_0 = \mathrm{SO}(2) \cdot \mathrm{G}_2$ it follows that, the identity components $(\mathrm{K}_t)_0 = \mathrm{H}_0 = \mathrm{S}^3$. Thus, $\mathrm{G}_t = \mathrm{SO}(4)$, or $\mathrm{Spin}(4) = \mathrm{S}^3 \times \mathrm{S}^3$. The latter, however, is impossible, since then $\mathrm{K}_t = \mathrm{S}^3 \times \mathbb{Z}_2 \triangleleft \mathrm{S}^3 \times \mathrm{Q}_8 = \mathrm{H}$ where Q_8 is the quaternion group of order 8. On the other hand, by Table 4.4 the slice representation at q is the natural tensor representation of $\mathrm{O}(2) \cdot \mathrm{G}_2$ on $\mathbb{R}^2 \otimes \mathbb{R}^7$, where the center $\mathbb{Z}_2 \subset \mathrm{Q}_8$ is in the kernel K_q and so in $\mathrm{K}_t \cap \mathrm{K}_q$. A contradiction. Therefore, $\mathrm{G}_t = \mathrm{SO}(4)$, and consequently, $\mathrm{G}_q = \mathrm{O}(2) \cdot \mathrm{G}_2$, $\mathrm{G}_r = \mathrm{O}(2) \cdot \mathrm{SU}(3)$ and $\mathrm{G}_{\ell_t} = \mathrm{SU}(3) \cdot \mathbb{Z}_2^2$.

By Lemma 6.3 we have $3 \leq \mathrm{rk}(\mathrm{G}) \leq 4$.

Case (i). Assume $\mathrm{rk}(\mathrm{G}) = 3$:

By Lemma 6.3 again $\dim M$ is even. By [BS] (table on page 219) $\mathrm{SO}(2) \cdot \mathrm{G}_2$ is not a subgroup in any rank 3 simple group. Therefore, $\mathrm{G} = \mathrm{L} \cdot \mathrm{G}_2$, where L is a rank one group. By Table 4.4 the face isotropy group $(\mathrm{G}_{\ell_r})_0 = \mathrm{SU}(2) \cdot \Delta(\mathrm{SO}(2))$ is diagonally embedded in $\mathrm{SO}(2) \cdot \mathrm{G}_2 \triangleleft \mathrm{G}_q \subset \mathrm{L} \cdot \mathrm{G}_2$. It follows that, the composition homomorphism $\mathrm{G}_t \subset \mathrm{G} \rightarrow \mathrm{L}$ is nontrivial, hence surjective onto L , because $\mathrm{G}_t = \mathrm{SO}(4)$. Hence $\mathrm{L} = \mathrm{SO}(3)$ and $\mathrm{G} = \mathrm{SO}(3) \cdot \mathrm{G}_2$ since the only proper nontrivial normal Lie subgroup of $\mathrm{SO}(4)$ is S^3 with quotient $\mathrm{SO}(3)$. By the above, we already know that, $\mathrm{G}_t = \mathrm{SO}(4)$ is a diagonal subgroup given by an epimorphism $\mathrm{SO}(4) \rightarrow \mathrm{SO}(3)$ and a monomorphism $\mathrm{SO}(4) \rightarrow \mathrm{G}_2$. It is clear that, up to conjugation, $\mathrm{G}_q = \mathrm{O}(2) \cdot \mathrm{G}_2 \subset \mathrm{G}$ where $\mathrm{O}(2) \subset \mathrm{SO}(3)$ is the standard upper 2×2 block matrices subgroup.

As in the proof of Proposition 7.2 we now claim that there is at most one polar action with the data as above. Since we are dealing with a non classical Lie group, however, we proceed as follows:

Given another C_3 type polar action of $\mathrm{G} = \mathrm{SO}(3) \cdot \mathrm{G}_2$ with isomorphic local data along a chamber C' with vertices t', q', r' . Without loss of generality we may assume that $\mathrm{G}_q = \mathrm{G}_{q'} \subset \mathrm{SO}(3) \cdot \mathrm{G}_2$, and moreover, $\mathrm{G}_t = \mathrm{G}_{t'}$ since any two $\mathrm{SO}(4)$ subgroups in G_2 are conjugate. Moreover, we can further assume that $\mathrm{G}_{\ell_t} = \mathrm{G}_{\ell_{t'}}$ since the singular isotropy groups pair for the slice representation at q is unique up to conjugation. In particular, the principal isotropy groups $\mathrm{H} = \mathrm{H}'$. We prove now $(\mathrm{G}_{\ell_q})_0 = (\mathrm{G}_{\ell_{q'}})_0 = \mathrm{SO}(2) \cdot \mathrm{SU}(2)$. This clearly implies the assertion since G_r is generated by $(\mathrm{G}_{\ell_q})_0$ and G_{ℓ_t} . Recall that $\mathrm{G}_t = \Delta(\mathrm{SO}(4)) \subset \mathrm{G}$, its composition with the projection to $\mathrm{G}_2 \triangleleft \mathrm{G}$ is a monomorphism, so is the composition of $(\mathrm{G}_{\ell_q})_0 \subset (\mathrm{G}_r)_0 = \mathrm{SO}(2) \cdot \mathrm{SU}(3)$ to G_2 , hence, $(\mathrm{G}_{\ell_q})_0$ is a diagonal subgroup of G_r , whose projection to the factor $\mathrm{SU}(3)$ is injective. Hence it suffices to show that the projection images of $(\mathrm{G}_{\ell_q})_0$ and $(\mathrm{G}_{\ell_{q'}})_0$ in $\mathrm{SU}(3) = (\mathrm{G}_{\ell_t})_0 = (\mathrm{G}_{\ell_{t'}})_0$ coincide. On the other hand, note that the projection image of $(\mathrm{G}_{\ell_q})_0$ in $\mathrm{SU}(3)$ is the normalizer $\mathrm{N}_0(\mathrm{H}_0)$ in $\mathrm{SU}(3) = (\mathrm{G}_{\ell_t})_0$, where H_0 is the identity component of the principal isotropy group. The above assertion follows.

As for existence we again note that $\mathrm{SO}(3) \mathbf{G}_2$ is a maximal subgroup of the isometry group \mathbf{F}_4 of the Cayley plane $\mathbb{O}\mathbb{P}^2$. The corresponding unique isometric action is indeed polar as proved in [GK] and of type \mathbf{C}_3 .

Case (ii). Assume $\mathrm{rk}(\mathbf{G}) = 4$:

By Lemma 2.7, $\dim M$ is odd. Consider the reduction $M^{\mathbf{H}_0}$ with the action of $\mathbf{N}_0(\mathbf{H}_0)$, the identity component of the normalizer. Note that, this is also a \mathbf{C}_3 type polar action, but the multiplicity triple is $(1, 1, 1)$. By appealing to Lemma 2.8, the codimension of $M^{\mathbf{H}_0}$ is divisible by 4. Thus from the $(1, 1, 1)$ case it follows that, the universal cover $\tilde{M}^{\mathbf{H}_0}$ is \mathbb{S}^{11} , and the identity component $\mathbf{N}_0(\mathbf{H}_0)$ is either $\mathrm{U}(3)$ or $\mathrm{SO}(3) \mathrm{SO}(4)$, modulo kernel.

We are going to prove that $M^{\mathbf{H}_0}$ is simply connected. It suffices to show that $M^{\mathbf{H}_0} \subset M$ is 2-connected. This follows trivially by the Connectivity Lemma of Wilking 2.6, if the codimension of $M^{\mathbf{H}_0}$ is at most 12.

If $\mathbf{G}_2 \triangleleft \mathbf{G}_q$ is a normal subgroup of \mathbf{G} , then $\mathbf{G} = \mathbf{L} \cdot \mathbf{G}_2$ where \mathbf{L} is a rank 2 group. Then $\mathbf{N}_0(\mathbf{H}_0)/\mathbf{H}_0$ is isomorphic to $\mathbf{L} \cdot \mathrm{SO}(3)$. Hence $\mathbf{L} = \mathrm{SO}(4)$. It is easy to count the codimension to see that it is strictly less than 12.

If \mathbf{G}_2 is not a normal subgroup, by Lemma 2.8 the isotropy representation of $\mathrm{SU}(3) \subset \mathbf{G}_2 \subset \mathbf{G}$ is spherical transitive. Hence, \mathbf{G} contains a normal simple Lie subgroup \mathbf{L} , such that $\mathbf{G}_2 \subset \mathrm{Spin}(7) \subset \mathbf{L}$ is *spherical*. We claim that $\mathbf{L} = \mathrm{Spin}(7)$. If not, \mathbf{L} contains $\mathrm{Spin}(8)$ such that $\mathrm{Spin}(7) \subset \mathbf{L}$ is a block subgroup in $\mathrm{Spin}(8)$, and hence $\mathbf{N}_0(\mathbf{H}_0)$ contains $\mathrm{Spin}(5)$, which contradicts the above. This proves that $\mathbf{G} = \mathbf{L}_1 \cdot \mathrm{Spin}(7)$, where \mathbf{L}_1 is a rank 1 group. From this we get that the isotropy subrepresentation of \mathbf{G}/\mathbf{H}_0 contains exactly three copies of the standard defining representation of $\mathrm{SU}(2)$, hence the desired estimate for the codimension.

In summary we conclude that $M^{\mathbf{H}_0} = \mathbb{S}^{11}$, $\mathbf{N}_0(\mathbf{H}_0) = \mathrm{SO}(3) \mathrm{SO}(4)$ and hence, from the multiplicity $(1, 1, 1)$ case, the chamber system for the action of $\mathbf{N}_0(\mathbf{H}_0)$ is a building of type \mathbf{C}_3 . By remark 3.6 we conclude that $\mathcal{C}(M, \mathbf{G})$ is a building. \square

PROPOSITION 8.2. *There is no polar action of type \mathbf{C}_3 type with multiplicities $(1, 1, 6)$, where the (effective) slice representation at T_q^\perp is the tensor product representation of $\mathrm{SO}(2) \mathrm{Spin}(7)$ on $\mathbb{R}^2 \otimes \mathbb{R}^8$.*

Proof. We will prove that, if there is such a slice representation at q , the chamber system $\mathcal{C}(M, \mathbf{G})$ is a building. The desired claim follows from the classification of \mathbf{C}_3 buildings, i.e., indeed there is no such a building.

To proceed, note that from Table 4.4 $\bar{\mathbf{G}}_q = \mathrm{SO}(2) \mathrm{Spin}(7)$, and the principal isotropy group $\bar{\mathbf{H}} = \mathrm{SU}(3)$. It follows that, up to local isomorphism $\mathbf{G}_t = \mathrm{SU}(3) \mathrm{SO}(3)$ with $\mathbf{K}_t = \mathrm{SU}(3)$. Notice that, the reduction $(M^{\mathbf{K}_t}, \mathbf{N}_0(\mathbf{K}_t))$ is of cohomogeneity 2 with the same section. It is clear that it is of type \mathbf{A}_3 since the q vertex is a vertex with angle $\pi/2$, because $\mathbf{N}_0(\mathbf{K}_t) \cap \mathbf{G}_q$ is \mathbf{T}^2 . By the classification of \mathbf{A}_3 geometries it follows that, $M^{\mathbf{K}_t}$ is either \mathbb{S}^8 or $\mathbb{R}\mathbb{P}^8$. We claim that $M^{\mathbf{K}_t} = \mathbb{S}^8$, and hence the chamber system for $(M^{\mathbf{K}_t}, \mathbf{N}_0(\mathbf{K}_t))$ is a building. By appealing to 3.6 it follows that $\mathcal{C}(M, \mathbf{G})$ is a building. To see the claim, it suffices to prove that $M^{\mathbf{K}_t}$ is orientable and hence simply connected, thanks to the positive curvature. By 2.8 the isotropy representation of $\mathbf{K}_t = \mathrm{SU}(3)$ is the defining complex representation. From this it is immediate that, $M^{\mathbf{K}_t} = M^{\mathbf{T}^2}$, and hence oriented, where $\mathbf{T}^2 \subset \mathbf{K}_t$ is a maximal torus. \square

PROPOSITION 8.3. *When the multiplicity triple is $(2, 2, 2)$, there are two scenarios. In either case $\mathcal{C}(M, \mathbf{G})$ is a building, with linear model the adjoint polar representation of either $\mathrm{SO}(7)$ or of $\mathrm{Sp}(3)$ on \mathbb{S}^{20} .*

Proof. By Lemma 2.5 we know that all vertex isotropy groups are connected. Notice that, by Table 4.4, the slice representation at q is the adjoint representation of $\mathrm{SO}(5)$ on \mathbb{R}^{10} . Together with Proposition 2.3, up to local isomorphism, the local isotropy group data are determined as follows: $\mathbf{G}_t = \mathrm{U}(3)$, $\mathbf{G}_q = \mathrm{SO}(5) \mathbf{S}^1$ and $\mathbf{G}_r = \mathrm{SO}(3) \mathrm{U}(2)$. Moreover, $\mathbf{H} = \mathbf{T}^3$, $\mathbf{G}_{\ell_t} = \mathrm{SO}(3) \mathrm{SO}(2) \mathbf{S}^1$, and $\mathbf{K}' = \mathrm{SO}(3) \triangleleft \mathbf{G}_{\ell_t}$.

Let $\mathrm{SO}(2) = \mathbf{K}' \cap \mathbf{H} \subset \mathbf{K}'$. Consider the reduction $(M^{\mathrm{SO}(2)}, \mathbf{N}(\mathrm{SO}(2)))$. It is once again a polar manifold with the same section. For such a reduction, notice that: the face ℓ_q has multiplicity 2, the face ℓ_t is exceptional with normal sphere \mathbb{S}^0 , and $\mathbf{G}_q \cap \mathbf{N}(\mathrm{SO}(2)) / \mathbf{G}_{\ell_t} \cap \mathbf{N}(\mathrm{SO}(2)) = \mathbb{S}^2$. Therefore, the action of $\mathbf{N}(\mathrm{SO}(2))$ is reducible with fundamental chamber $rq q'$, where q' is a reflection image of q , and $\overline{r q} = \ell_t$ is of exceptional orbit type. In particular, the multiplicities at q' are $(2, 2)$, hence the slice representation at q' for the $\mathbf{N}(\mathrm{SO}(2))$ -action is again the adjoint representation of $\mathrm{SO}(5)$ on \mathbb{R}^{10} . This clearly implies that q' is a fixed point. On the other hand, notice that $M^{\mathrm{SO}(2)}$ is orientable and hence simply connected. Therefore, by Theorem 6.2 of [FGT] we know that $M^{\mathrm{SO}(2)} = \mathbb{S}^{10}$. Since Property (P) holds for $\mathrm{SO}(2)$ it follows from Remark 3.6 that $\mathcal{C}(M, \mathbf{G})$ is a building. \square

Remark 8.4. We remark that in the above proof, the chamber system of $(M^{\mathrm{SO}(2)}, \mathbf{N}(\mathrm{SO}(2)))$ is a building of type $\mathbf{A}_1 \times \mathbf{C}_2$ but the one for $(M^{\mathrm{SO}(2)}, \mathbf{N}_0(\mathrm{SO}(2)))$ is not.

PROPOSITION 8.5. *In the case of the multiplicities $(4, 4, 5)$, the chamber system $\mathcal{C}(M; \mathbf{G})$ is covered by a building, with the isotropy representation of $\mathrm{SO}(14) / \mathrm{U}(7)$ as a linear model.*

Proof. By Lemma 2.5 we know that all isotropy groups are connected. Note that $\bar{\mathbf{G}}_t = \mathrm{Sp}(3)$, and $\bar{\mathbf{G}}_q = \mathrm{SU}(5)$ or $\mathrm{U}(5)$. By Lemma 2.3, it is easy to see that:

- if \mathbf{G}_t is semisimple, then, up to local isomorphism, $\mathbf{G}_t = \mathrm{Sp}(3)$, $\mathbf{G}_r = \mathrm{Sp}(2) \mathrm{SU}(3)$, $\mathbf{G}_q = \mathrm{SU}(5) \mathrm{Sp}(1)$ and $\mathbf{G}_{\ell_t} = \mathrm{SU}(3) \mathrm{Sp}(1)^2$, where $\mathrm{Sp}(1) = \mathbf{K}_q$ is a subgroup of \mathbf{G}_t .
- if \mathbf{G}_t is not semisimple, then $\mathbf{K}_t = \mathbf{S}^1$, and all isotropy groups data are the product of \mathbf{S}^1 with the corresponding data above.

We now prove that \mathbf{G} contains $\mathrm{SU}(7)$ as a normal subgroup. By Lemma 2.8 the isotropy representations of $\mathbf{G} / \mathrm{Sp}(2)$ and $\mathbf{G} / \mathrm{SU}(3)$ are both spherical, where $\mathrm{Sp}(2)$, $\mathrm{SU}(3)$ are normal factors of face isotropy groups. Hence, a normal factor \mathbf{L} of \mathbf{G} is either $\mathrm{SO}(n)$ or $\mathrm{SU}(n)$, by Table B in [GWZ]. Moreover, the subgroup $\mathbf{K}_q \subset \mathbf{G}_t$ is contained in a block subgroup $\mathrm{SO}(4) \subset \mathbf{L}$ (resp. a block subgroup $\mathrm{SU}(2) \subset \mathbf{L}$) if $\mathbf{L} = \mathrm{SO}(n)$ (resp. $\mathbf{L} = \mathrm{SU}(n)$). Since $\mathbf{N}_0(\mathbf{K}_q)$ contains \mathbf{G}_q , it follows that $n \geq 14$ (resp. $n \geq 7$) if $\mathbf{L} = \mathrm{SO}(n)$ (resp. $\mathrm{SU}(n)$). To rule out the former case, consider the fixed point set $M^{\mathbf{K}_q}$ with the polar action of $\mathbf{N}_0(\mathbf{K}_q)$. It is clearly a reducible cohomogeneity 2 action with q a vertex of angle $\pi/4$. By the Dual Generation Lemma 7.2 of [FGT] it follows that $\mathbf{N}_0(\mathbf{K}_q)$ is either \mathbf{G}_q (the fixed point case) or the product of $\mathrm{SU}(5) \triangleleft \mathbf{G}_q$ with the face isotropy group opposite to q in the reduction $M^{\mathbf{K}_q} / \mathbf{N}_0(\mathbf{K}_q)$. From this it is immediate that $\mathbf{L} = \mathrm{SU}(7)$.

Note that if \mathbf{G}_t is semisimple, or $\dim(M)$ is even, then $\mathrm{rank} \mathbf{G} \leq 6$, by the Rank Lemma, and hence $\mathbf{G} = \mathrm{SU}(7)$. For the remaining case, i.e., $\dim(M)$ being odd and $\mathbf{G}_t = \mathbf{S}^1 \cdot \mathrm{Sp}(3)$,

we now prove that $G = U(7)$, up to local isomorphism. Indeed, it is clear that $\text{rank } G = 7$, and hence $G = \text{SU}(7) \cdot L_2$, where L_2 is a rank 1 group. It suffices to prove that $L_2 = S^1$. Let $K' = \text{SU}(3) \triangleleft G_{\ell_t}$. It is clear that the projection $p_2 : G \rightarrow L_2$ is trivial, when restricted to either of $\text{Sp}(3) \triangleleft G_t$ and $K' \subset G_q$. By primitivity 2.2, $G = \langle G_t, G_{\ell_t} \rangle = \langle G_t, K' \rangle$. Therefore, $p_2(G_t) = L_2$ and hence, $L_2 = S^1$.

To complete the proof, we split into two cases, i.e, $\dim(M)$ being even or odd. For the former, $K_t = S^1$ and $G = \text{SU}(7)$. It is clear that $G_t = \text{Sp}(3) S^1$ is a subgroup of $U(6) \subset \text{SU}(7)$ and $G_q = \text{SU}(5) \text{Sp}(1) \cdot S^1$ is the normalizer $N(\text{Sp}(1))$ in G , where $\text{Sp}(1) \triangleleft G_{\ell_r} \subset G_t$. This forces all isotropy groups data to be the same as for the linear cohomogeneity 2 polar action on \mathbb{CP}^{20} induced from the isotropy representation of $\text{SO}(14)/U(7)$. Hence, in particular, the chamber system $\mathcal{C}(M; G)$ is covered by a building. For the latter, $G = \text{SU}(7)$ or $U(7)$ depending on $K_t = \{1\}$ or S^1 . The fixed point set $M^{K'}$ is odd dimensional, since the isotropy representation of K' is the defining complex representation. Note that $N_0(K') = \text{SU}(4) T^i \cdot K'$, $i = 1, 2$, and $M^{K'}$ is equivariantly diffeomorphic to \mathbb{S}^{11} with a standard linear cohomogeneity one action of type C_2 . Hence, by Lemma 3.4, $\mathcal{C}(M; G)$ is a building. \square

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